Error estimates in periodic homogenization with a non-homogeneous Dirichlet condition.

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Abstract

In this paper we investigate the homogenization problem with a non-homogeneous Dirichlet condition. Our aim is to give error estimates with boundary data in $H^{1/2}(\partial\Omega)$. The tools used are those of the unfolding method in periodic homogenization.

1 Introduction

We consider the following homogenization problem:

$$\phi^\varepsilon \in H^1(\Omega), \qquad -\mathrm{div}\big(A_\varepsilon \nabla \phi^\varepsilon\big) = f \qquad \text{in} \quad \Omega, \qquad \phi^\varepsilon = g \qquad \text{on} \quad \partial\Omega \tag{1.1}$$

where A_ε is a periodic matrix satisfying the usual condition of uniform ellipticity and where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$[1]. We know (see e.g. [4], [10], [13]) that the function ϕ^ε weakly converges in $H^1(\Omega)$ towards the solution Φ of the homogenized problem

$$\Phi \in H^1(\Omega), \qquad -\mathrm{div}\big(\mathcal{A}\nabla \Phi\big) = f \qquad \text{in} \quad \Omega, \qquad \Phi = g \qquad \text{on} \quad \partial\Omega \tag{1.2}$$

where $\mathcal{A}$ is the homogenized matrix (see (4.4) and (4.5)). Using the results in [10] we can give an approximation of ϕ^ε belonging to $H^1(\Omega)$ and we easily obtain

$$\phi^\varepsilon - \Phi - \varepsilon \sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big) \longrightarrow 0 \quad \text{strongly in} \quad H^1(\Omega)$$

where $\mathcal{Q}_\varepsilon$ is the *scale-splitting operator* (see [10] or Subsection 2.4) and where the χ_i are the correctors (see (4.2)).

One of the aim of this paper is to give error estimates for this homogenization problem. Obviously, if we have $g \in H^{3/2}(\partial\Omega)$ and the appropriate assumptions on

[1] The homogenization problem with L^p boundary data is investigated in [3].

the boundary of the domain then we can apply the results in [4], [13], [14], [15], [16] and [22] to deduce error estimates. All of them require that the function Φ belongs at least to $H^2(\Omega)$. Here, the solution Φ of the homogenized problem (1.2) is only in $H^1(\Omega)\cap H^2_{loc}(\Omega)$. In this paper we have to deal with this lack of regularity; this is the main difficulty.

The tools of the unfolding method in periodic homogenization to obtain error estimates (see [14], [15] and [16]) are the projection theorems. This is why we prove two new projection theorems; the Theorems 3.1 and 3.2. Here, both theorems concern the functions $\phi\in H^1_0(\Omega)$ satisfying $\nabla\phi/\rho\in L^2(\Omega;\mathbb{R}^n)$ where $\rho(x)$ is the distance between x and the boundary of Ω. In the first one we give the distance between $\mathcal{T}_\varepsilon(\phi)$ (see [10] or Subsection 2.4.1 for the definition of the unfolding operator $\mathcal{T}_\varepsilon$) and the space $L^2(\Omega;H^1_{per}(Y))$ in terms of the L^2 norms of ϕ/ρ and $\nabla\phi/\rho$ and obviously ε. In the second one we prove an upper bound for the distance between $\mathcal{T}_\varepsilon(\nabla\phi)$ and the space $\nabla H^1(\Omega)\oplus\nabla_y L^2(\Omega;H^1_{per}(Y))$; again in terms of the L^2 norms of ϕ/ρ and $\nabla\phi/\rho$ and ε (see Section 3). This last theorem is partially a consequence of the first one. In this paper we derive the new error estimates from the second projection theorem and those obtained in [16].

Different results are known about the global H^1 error estimate regarding the classical homogenization problem (1.1) (see e.g. [4], [13]). Those with the minimal assumptions are given in [15]; if the solution of the homogenized problem (1.2) belongs to $H^2(\Omega)$ -see Proposition 4.3 in [15]- (respectively $H^{3/2}(\Omega)$; see Theorem 3.3 in [16]) then the H^1 global error is of order $\varepsilon^{1/2}$ (resp. $\varepsilon^{1/4}$) while if this solution belongs to $H^2_{loc}(\Omega)\cap W^{1,p}(\Omega)$ $(p>2)$ the obtained H^1 global error is smaller and depends on p (see Proposition 4.4 in [15])[2]. Here, with a non-homogeneous Dirichlet condition belonging only to $H^{1/2}(\partial\Omega)$ we do not obtain a global H^1 error estimate. The L^2 global error estimate only requires a boundary of Ω sufficiently smooth (of class $\mathcal{C}^{1,1}$) or a convex open set. Obviously if it is possible to make use of a global H^1 error estimate, the L^2 global error will be better (the reader will be able to compare the Theorem 3.2 in [16] with the Theorem 6.3). The H^1 local error estimate is always linked to the L^2 global error and never needs more assumption (see Theorem 3.2 in [16] or the proof of Theorem 6.1).

The paper is organized as follows. In Section 2 we introduce a few general notations, then we give some reminds[3] on lemmas, definitions and results about the unfolding method in periodic homogenization (see [10]), then we prove some new results involving the main operators of this method. Section 3 is devoted to the new projection theorems. In Section 4, we recall the main results on the classical homogenization problem. In Section 5 we introduce an operator which allows to lift the distributions belonging to $H^{-1/2}(\partial\Omega)$ in functions belonging to $L^2(\Omega)$; this lifting operator will play an important role in the case of strongly oscillating boundary data. In Section 6 we derive the error estimates results (Theorems 6.1 and 6.3) with a non-homogenous Dirichlet condition. We

[2] These propositions or theorem are proved with a Dirichlet condition, with a non-homogenous Dirichlet condition belonging to $H^{3/2}(\partial\Omega)$ the results are obviously the same.

[3] We want to simplify the reading to a non-familiar reader with the unfolding method

end the paper by investigating a case where the boundary data are strongly oscillating (see Theorem 7.1 in Section 7). A forthcoming paper will be devoted to homogenization problems with other strongly oscillating boundary data.

As general references on the homogenization theory we refer to [1], [4] and [13]. The reader is referred to [10], [12] and [13] for an introduction of the unfolding method in periodic homogenization. The following papers [5], [6], [7], [8], [11], [19], [24] give various applications of the unfolding method in periodic homogenization. As far as the error estimates are concerned, we refer to [2], [4], [14], [15], [16], [20], [22] and [23].

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2 Preliminaries

2.1 Notations

• The space $\mathbb{R}^k$ ($k \geq 1$) is endowed with the standard basis $\big(\mathbf{e}_1, \dots, \mathbf{e}_k\big)$; the euclidian norm is denoted $|\cdot|$.
• We denote by Ω a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary.[4] Let $\rho(x)$ be the distance between $x \in \mathbb{R}^n$ and the boundary of Ω, we set

$$\widetilde{\Omega}_\gamma = \Big\{x \in \Omega \mid \rho(x) < \gamma\Big\} \qquad \widetilde{\widetilde{\Omega}}_\gamma = \Big\{x \in \mathbb{R}^n \mid \rho(x) < \gamma\Big\} \qquad \gamma \in \mathbb{R}^{*+}.$$

• There exist constants a, A and γ_0 strictly positive and $M \geq 1$, a finite number N of local euclidian coordinate systems $(O_r; \mathbf{e}_{1r}, \dots, \mathbf{e}_{nr})$ and mappings $f_r : [-a, a]^{n-1} \longrightarrow \mathbb{R}$, Lipschitz continuous with ratio M, $1 \leq r \leq N$, such that (see e.g. [17] or [18])

$$\begin{aligned}
&\partial\Omega = \bigcup_{r=1}^{N} \Big\{x = x_r^{'} + x_{nr}\mathbf{e}_{nr} \in \mathbb{R}^n \ \mid\ x_r^{'} \in \Delta_a \ \text{ and } \ x_{nr} = f_r(x_r^{'})\Big\},\\
&\text{where } \ x_r^{'} = x_{1r}\mathbf{e}_{1r} + \dots + x_{n-1r}\mathbf{e}_{n-1r}, \quad \Delta_a = \Big\{x_r^{'} \mid x_{ir} \in]-a, a[, \ i \in \{1, \dots, n-1\}\Big\}\\
&\widetilde{\Omega}_{\gamma_0} \subset \bigcup_{r=1}^{N} \Omega_r \subset \Omega, \qquad \Omega_r = \Big\{x \in \mathbb{R}^n \mid x_r^{'} \in \Delta_a \text{ and } f_r(x_r^{'}) < x_{nr} < f_r(x_r^{'}) + A\Big\}\\
&\widetilde{\widetilde{\Omega}}_{\gamma_0} \subset \bigcup_{r=1}^{N} \Big\{x \in \mathbb{R}^n \mid x_r^{'} \in \Delta_a \ \text{ and } \ f_r(x_r^{'}) - A < x_{nr} < f_r(x_r^{'}) + A\Big\}\\
&\forall r \in \{1, \dots, N\}, \quad \forall x \in \Omega_r \quad \text{we have} \quad \frac{1}{2M}(x_{nr} - f_r(x_r^{'})) \leq \rho(x) \leq x_{nr} - f_r(x_r^{'}).
\end{aligned} \tag{2.1}$$

[4]In Section 5 and those which follow, we will assume that Ω is a bounded domain of class $\mathcal{C}^{1,1}$ or an open bounded convex set.

• We set

$$Y =]0,1[^n, \qquad \Xi_\varepsilon = \Big\{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi + Y) \subset \Omega\Big\},$$
$$\widehat{\Omega}_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\Big), \qquad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon,$$

where ε is a strictly positive real.

• We define

⋆ $H^1_\rho(\Omega) = \Big\{\phi \in L^2(\Omega) \mid \rho\nabla\phi \in L^2(\Omega;\mathbb{R}^n)\Big\}$,

⋆ $L^2_{1/\rho}(\Omega) = \Big\{\phi \in L^2(\Omega) \mid \phi/\rho \in L^2(\Omega)\Big\}$,

⋆ $H^1_{1/\rho}(\Omega) = \Big\{\phi \in H^1_0(\Omega) \mid \nabla\phi/\rho \in L^2(\Omega;\mathbb{R}^n)\Big\}$.

We endow $H^1_\rho(\Omega)$ (resp. $H^1_{1/\rho}(\Omega)$) with the norm

$$\forall \phi \in H^1_\rho(\Omega), \quad ||\phi||_\rho = ||\phi||_{L^2(\Omega)} + ||\rho\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)}$$
$$(\text{ resp.} \quad \forall\phi \in H^1_{1/\rho}(\Omega), \quad ||\phi||_{1/\rho} = \big\|\nabla\phi/\rho\big\|_{L^2(\Omega;\mathbb{R}^n)}).$$

Note that if ϕ belongs to $H^1_\rho(\Omega)$ then the function $\psi = \rho\phi$ is in $H^1_0(\Omega)$ and vice versa if a function ψ belongs to $H^1_0(\Omega)$ then $\phi = \psi/\rho$ is in $H^1_\rho(\Omega)$ since we have (see [9] or [21])

$$\forall\psi \in H^1_0(\Omega), \qquad \big\|\psi/\rho\big\|_{L^2(\Omega)} \le C||\nabla\psi||_{L^2(\Omega;\mathbb{R}^n)}. \tag{2.2}$$

Below we recall a classical extension lemma which is proved for example in [15] or which can be proved using the local charts (2.1).

Lemma 2.1. *Let Ω be a bounded domain with a Lipschitz boundary, there exist $c_0 \ge 1$ (which depends only on the boundary of Ω) and a linear and continuous extension operator $\mathcal{P}$ from $L^2(\Omega)$ into $L^2(\mathbb{R}^n)$ which also maps $H^1(\Omega)$ into $H^1(\mathbb{R}^n)$ such that*

$$\begin{aligned}\forall\phi \in L^2(\Omega), \quad \mathcal{P}(\phi)_{|\Omega} = \phi, \qquad & ||\mathcal{P}(\phi)||_{L^2(\mathbb{R}^n)} \le C||\phi||_{L^2(\Omega)},\\ & ||\mathcal{P}(\phi)||_{L^2(\widetilde{\widetilde{\Omega}}_\gamma)} \le C||\phi||_{L^2(\widetilde{\Omega}_{c_0\gamma})}.\end{aligned} \tag{2.3}$$

Moreover we have

$$\forall\phi \in H^1(\Omega), \qquad ||\nabla\mathcal{P}(\phi)||_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le C||\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)}.$$

From now on, if need be, a function ϕ belonging to $L^2(\Omega)$ (resp. $H^1(\Omega)$) will be extended to a function belonging to $L^2(\mathbb{R}^n)$ (resp. $H^1(\mathbb{R}^n)$) using the above lemma. The extension will be still denoted ϕ.

2.2 A characterization of the functions belonging to $H^1_{1/\rho}(\Omega)$

The two first projection theorems (see [15]) regarded the functions belonging to $H^1_0(\Omega)$ while those in [16] regarded the functions in $H^1(\Omega)$. In this paper we prove two new projection theorems which involve the functions in $H^1_{1/\rho}(\Omega)$; this is why we first give a simple characterization of these functions in the Lemma 2.2 below.

Observe first that if a function ϕ satisfies $\phi/\rho \in H^1_0(\Omega)$ then ϕ belongs to $H^1_{1/\rho}(\Omega)$. The reverse is true.

Lemma 2.2. *Let Ω be a bounded domain with a Lipschitz boundary, we have*

$$\phi \in H^1_{1/\rho}(\Omega) \iff \phi/\rho \in H^1_0(\Omega).$$

Furthermore there exists a constant which depends only on $\partial\Omega$ such that

$$\forall \phi \in H^1_{1/\rho}(\Omega) \qquad \big\|\phi/\rho^2\big\|_{L^2(\Omega)} + \big\|\phi/\rho\big\|_{H^1(\Omega)} \le C||\phi||_{1/\rho}. \tag{2.4}$$

Proof. Step 1. Let ϕ be in $H^1(]-a,a[^{n-1}\times]0,A[)$ $(a,\ A>0)$ satisfying $\dfrac{1}{x_n}\nabla\phi(x) \in L^2(]-a,a[^{n-1}\times]0,A[)$ and $\phi(x)=0$ for a.e. x in $]-a,a[^{n-1}\times\{0\}\cup]-a,a[^{n-1}\times\{A\}$. We have

$$\int_{]-a,a]^{n-1}\times]0,A[} \frac{|\phi(x)|^2}{x_n^4}dx \le \frac{1}{2}\int_{]-a,a]^{n-1}\times]0,A[} \frac{|\nabla\phi(x)|^2}{x_n^2}dx. \tag{2.5}$$

To prove (2.5), we choose $\eta>0$ and we integrate by parts $\displaystyle\int_{]-a,a]^{n-1}\times]0,A[} \frac{|\phi(x)|^2}{(\eta+x_n)^4}dx$, then thanks to the identity relation $2bc \le b^2+c^2$ we obtain

$$\begin{aligned}\int_{]-a,a]^{n-1}\times]0,A[} \frac{|\phi(x)|^2}{(\eta+x_n)^4}dx &\le \frac{1}{2}\int_{]-a,a]^{n-1}\times]0,A[} \frac{1}{(\eta+x_n)^2}\Big|\frac{\partial\phi}{\partial x_n}(x)\Big|^2 dx\\ &\le \frac{1}{2}\int_{]-a,a]^{n-1}\times]0,A[} \frac{|\nabla\phi(x)|^2}{x_n^2}dx.\end{aligned}$$

Passing to the limit $(\eta\to 0)$ it leads to (2.5).

Step 2. Let h be in $W^{1,\infty}(\Omega)$ such that

$$\forall x\in\Omega, \qquad \begin{aligned}&h(x)\in[0,1],\\ &h(x)=1 \quad \text{if} \quad \rho(x)\ge\gamma_0,\\ &h(x)=0 \quad \text{if} \quad \rho(x)\le\gamma_0/2.\end{aligned}$$

Let ϕ be in $H^1_{1/\rho}(\Omega)$. The function $\phi h/\rho^4$ belongs to $H^1_0(\Omega)$, therefore as a consequence of the Poincaré's inequality we obtain

$$\begin{aligned}\int_\Omega \frac{|\phi(x)h(x)|^2}{\rho(x)^4}dx &\le C\int_\Omega \Big|\nabla\Big(\frac{\phi(x)h(x)}{\rho(x)^4}\Big)\Big|^2 dx \le C\int_\Omega \big(|\nabla\phi(x)|^2+|\phi(x)|^2\big)dx\\ &\le C\int_\Omega |\nabla\phi(x)|^2 dx \le C\int_\Omega \frac{|\nabla\phi(x)|^2}{\rho(x)^2}dx.\end{aligned} \tag{2.6}$$

Then using the local chart of Ω_r given by (2.1), the inequality (2.5) and thanks to a simple change of variables we get

$$\int_{\Omega_r} \frac{|\phi(x)\big(1-h(x)\big)|^2}{\rho(x)^4} dx \le C \int_{\Omega_r} \frac{|\nabla\big(\phi(x)(1-h(x)\big)|^2}{\rho(x)^2} dx \le C \int_{\Omega_r} \frac{|\nabla\phi(x)|^2+|\phi(x)|^2}{\rho(x)^2} dx.$$

Since $\phi \in H^1_0(\Omega)$ the function ϕ/ρ belongs to $L^2(\Omega)$ and we have (2.2). Hence, adding these inequalities $(r=1,\dots,N)$ we obtain

$$\int_{\Omega} \frac{|\phi(x)\big(1-h(x)\big)|^2}{\rho(x)^4} dx \le C \int_{\Omega} \frac{|\nabla\phi(x)|^2}{\rho(x)^2} dx. \tag{2.7}$$

Finally $\phi/\rho^2 \in L^2(\Omega)$ and (2.6)-(2.7) lead to $\big\|\phi/\rho^2\big\|_{L^2(\Omega)} \le C||\phi||_{1/\rho}$ and then (2.4). $\square$

2.3 Two lemmas

In the Lemma 2.3 we give sharp estimates of a function on the boundary and in a neighborhood of the boundary of Ω. The second estimate in (2.8) is used to obtain the L^2 global error.

Lemma 2.3. *Let Ω be a bounded domain with a Lipschitz boundary, there exists $\gamma_0 > 0$ (see Subsection 2.2) such that for any $\gamma \in]0,\gamma_0]$ and for any $\phi \in H^1(\Omega)$ we have*

$$\begin{aligned} ||\phi||_{L^2(\partial\Omega)} &\le \frac{C}{\gamma^{1/2}}\big(||\phi||_{L^2(\widetilde{\Omega}_\gamma)} + \gamma||\nabla\phi||_{L^2(\widetilde{\Omega}_\gamma;\mathbb{R}^n)}\big),\\ ||\phi||_{L^2(\widetilde{\Omega}_\gamma)} &\le C\big(\gamma^{1/2}||\phi||_{L^2(\partial\Omega)} + \gamma||\nabla\phi||_{L^2(\widetilde{\Omega}_\gamma;\mathbb{R}^n)}\big). \end{aligned} \tag{2.8}$$

The constants do not depend on γ.

Proof. Let ψ be in $H^1(]-a,a[^{n-1}\times]0,A[)$. For $\eta \in]0,A[$ we have

$$\begin{aligned} ||\psi||^2_{L^2(]-a,a[^{n-1}\times\{0\})} &\le \frac{C}{\eta}||\psi||^2_{L^2(]-a,a[^{n-1}\times]0,\eta[)} + C\eta||\nabla\psi||^2_{L^2(]-a,a[^{n-1}\times]0,\eta[;\mathbb{R}^n)},\\ ||\psi||^2_{L^2(]-a,a[^{n-1}\times]0,\eta[)} &\le C\eta||\psi||^2_{L^2(]-a,a[^{n-1}\times\{0\})} + C\eta^2||\nabla\psi||^2_{L^2(]-a,a[^{n-1}\times]0,\eta[;\mathbb{R}^n)}. \end{aligned}$$

The constants do not depend on η. Now, let ϕ be in $H^1(\Omega)$. We use the above estimates, the local charts of $\widetilde{\Omega}_{\gamma_0}$ given by (2.1) and a simple change of variables to obtain (2.8). $\square$

In this second lemma we show that a function in $H^1_0(\Omega)$ can be approached by functions vanishing close to the boundary of Ω. Among other things this lemma is used to give an approximation of ϕ via the scale-splitting operator $\mathcal{Q}_\varepsilon$ (see Lemma 2.6) and it is also used in the main projection theorem (Theorem 3.2).

Lemma 2.4. *Let ϕ be in $H^1_0(\Omega)$, there exists $\phi_\varepsilon \in H^1(\mathbb{R}^n)$ satisfying*

$$\begin{aligned}
&\phi_\varepsilon(x) = 0 \qquad \text{for a.e. } x \notin \widetilde{\Omega}_{6\sqrt{n}\varepsilon},\\
&||\phi - \phi_\varepsilon||_{L^2(\Omega)} \le C\varepsilon||\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)}, \qquad ||\phi_\varepsilon||_{H^1(\Omega)} \le C||\phi||_{H^1(\Omega)}.
\end{aligned} \tag{2.9}$$

Moreover, if $\phi \in H^1_{1/\rho}(\Omega)$ then we have

$$\big\|(\phi - \phi_\varepsilon)/\rho\big\|_{L^2(\Omega)} \le C\varepsilon||\nabla\phi||_{1/\rho}, \qquad ||\phi_\varepsilon||_{1/\rho} \le C||\phi||_{1/\rho}. \tag{2.10}$$

The constant C is independent of ε.

Proof. Let ϕ be in $H^1_0(\Omega)$. We define ϕ_ε by

$$\phi_\varepsilon(x) = \begin{cases} \dfrac{(\rho(x) - 6\sqrt{n}\varepsilon)^+}{\rho(x)}\phi(x) & \text{for a. e.} \quad x \in \Omega,\\ 0 & \text{for a. e.} \quad x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

where $\delta^+ = \max\{0, \delta\}$. The above function ϕ_ε belongs to $H^1(\mathbb{R}^n)$ and satisfies $\phi_\varepsilon = 0$ outside $\widetilde{\Omega}_{6\sqrt{n}\varepsilon}$. Then due to the fact that ϕ/ρ belongs to $L^2(\Omega)$ and verifies $\|\phi/\rho\|_{L^2(\Omega)} \le C||\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)}$ we obtain the estimates in (2.9). If $\phi \in H^1_{1/\rho}(\Omega)$ we use the estimate (2.4) to obtain (2.10). $\square$

2.4 Reminds and complements on the unfolding operators

In the sequel, we will make use of some definitions and results from [10] concerning the periodic unfolding method. Below we remind them briefly.

2.4.1 Some reminds

For almost every $x \in \mathbb{R}^n$, there exists an unique element in $\mathbb{Z}^n$ denoted $[x]$ such that

$$x = [x] + \{x\}, \qquad \{x\} \in Y.$$

• *The unfolding operator $\mathcal{T}_\varepsilon$.*
For any $\phi \in L^1(\Omega)$, the function $\mathcal{T}_\varepsilon(\phi) \in L^1(\Omega \times Y)$ is given by

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon y\Big) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y,\\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \tag{2.11}$$

Since $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$, using Proposition 2.5 in [10] we get

$$\Big|\int_\Omega \phi(x)dx - \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi)(x, y)dxdy\Big| \le \int_{\Lambda_\varepsilon} |\phi(x)|dx \le ||\phi||_{L^1(\widetilde{\Omega}_{\sqrt{n}\varepsilon})} \tag{2.12}$$

For $\phi \in L^2(\Omega)$ we have
$$||\mathcal{T}_\varepsilon(\phi)||_{L^2(\Omega)} \le ||\phi||_{L^2(\Omega)}. \tag{2.13}$$
We also have (see Proposition 2.5 in [10]) for $\phi \in H^1(\Omega)$ (resp. $\psi \in H^1_0(\Omega)$)
$$\begin{aligned} &||\mathcal{T}_\varepsilon(\phi) - \phi||_{L^2(\widehat{\Omega}_\varepsilon \times Y)} \le C\varepsilon ||\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)} \\ (\text{ resp.} \quad &||\mathcal{T}_\varepsilon(\psi) - \psi||_{L^2(\Omega \times Y)} \le C\varepsilon ||\nabla\psi||_{L^2(\Omega;\mathbb{R}^n)} \quad). \end{aligned} \tag{2.14}$$

• *The local average operator* $\mathcal{M}_\varepsilon$
For $\phi \in L^1(\mathbb{R}^n)$, the function $\mathcal{M}_\varepsilon(\phi) \in L^\infty(\mathbb{R}^n)$ is defined by
$$\mathcal{M}_\varepsilon(\phi)(x) = \int_Y \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) dy \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{2.15}$$
The value of $\mathcal{M}_\varepsilon(\phi)$ in the cell $\varepsilon(\xi + Y)$ ($\xi \in \mathbb{Z}^n$) will be denoted $\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)$. In [10] we proved the following results:
For $\phi \in L^2(\Omega)$ we have
$$||\mathcal{M}_\varepsilon(\phi)||_{L^2(\Omega)} \le C||\phi||_{L^2(\Omega)}, \qquad ||\mathcal{M}_\varepsilon(\phi) - \phi||_{H^{-1}(\Omega)} \le C\varepsilon||\phi||_{L^2(\Omega)} \tag{2.16}$$
and for $\psi \in H^1_0(\Omega)$ (resp. $\phi \in H^1(\Omega)$) we have
$$\begin{aligned} &||\mathcal{M}_\varepsilon(\psi) - \psi||_{L^2(\Omega)} \le C\varepsilon||\nabla\psi||_{L^2(\Omega;\mathbb{R}^n)} \\ (\text{resp.} \quad &||\mathcal{M}_\varepsilon(\phi) - \phi||_{L^2(\widehat{\Omega}_\varepsilon)} \le C\varepsilon||\nabla\phi||_{L^2(\Omega;\mathbb{R}^n)} \ ). \end{aligned} \tag{2.17}$$

• *The scale-splitting operator* $\mathcal{Q}_\varepsilon$.
⋆ For $\phi \in L^1(\mathbb{R}^n)$, the function $\mathcal{Q}_\varepsilon(\phi) \in W^{1,\infty}(\mathbb{R}^n)$ is given by
$$\mathcal{Q}_\varepsilon(\phi)(x) = \sum_{\xi \in \mathbb{Z}^n} \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) H_{\varepsilon,\xi}(x) \qquad \text{for a.e. } x \in \mathbb{R}^n,$$
where
$$\begin{aligned} &H_{\varepsilon,\xi}(x) = H\Big(\frac{x - \varepsilon\xi}{\varepsilon}\Big) \qquad \text{with} \\ &H(z) = \begin{cases} \big(1 - |z_1|\big)\big(1 - |z_2|\big)\dots\big(1 - |z_n|\big) & \text{if} \quad z \in [-1,1]^n, \\ 0 & \text{if} \quad z \in \mathbb{R}^n \setminus [-1,1]^n. \end{cases} \end{aligned}$$
Below, we remind some results about $\mathcal{Q}_\varepsilon$ proved in [10] and [16].
⋆ For $\phi \in L^2(\mathbb{R}^n)$ we have
$$||\mathcal{Q}_\varepsilon(\phi)||_{L^2(\mathbb{R}^n)} \le C||\phi||_{L^2(\mathbb{R}^n)}, \qquad ||\nabla\mathcal{Q}_\varepsilon(\phi)||_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le \frac{C}{\varepsilon}||\phi||_{L^2(\mathbb{R}^n)} \tag{2.18}$$
and
$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \qquad \text{strongly in} \quad L^2(\mathbb{R}^n).$$

$\star$ For $\phi \in H^1(\mathbb{R}^n)$ we have

$$\begin{aligned} ||\nabla \mathcal{Q}_\varepsilon(\phi)||_{L^2(\mathbb{R}^n;\mathbb{R}^n)} &\leq C||\nabla\phi||_{L^2(\mathbb{R}^n;\mathbb{R}^n)}, \\ ||\phi - \mathcal{Q}_\varepsilon(\phi)||_{L^2(\mathbb{R}^n)} &\leq C\varepsilon||\nabla\phi||_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \end{aligned} \tag{2.19}$$

and

$$\mathcal{Q}_\varepsilon(\phi) \longrightarrow \phi \qquad \text{strongly in} \qquad H^1(\mathbb{R}^n). \tag{2.20}$$

$\star$ For $\phi \in L^2(\mathbb{R}^n)$ and $\chi \in L^2(Y)$ we have $\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\dfrac{\cdot}{\varepsilon}\Big\}\Big) \in L^2(\mathbb{R}^n)$, $\nabla\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\dfrac{\cdot}{\varepsilon}\Big\}\Big) \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned} \big\|\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\mathbb{R}^n)} &\leq C\|\phi\|_{L^2(\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \big\|\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\widetilde{\Omega}_{\sqrt{n}\varepsilon})} &\leq C\|\phi\|_{L^2(\widetilde{\widetilde{\Omega}}_{3\sqrt{n}\varepsilon})}\|\chi\|_{L^2(Y)}. \end{aligned} \tag{2.21}$$

Moreover, if $\phi \in H^1(\mathbb{R}^n)$ then we have

$$\begin{aligned} \big\|\big(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi)\big)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\mathbb{R}^n)} &\leq C\varepsilon\|\nabla\phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \big\|\nabla\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} &\leq C\|\nabla\phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \\ \big\|\nabla\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\widetilde{\Omega}_{\sqrt{n}\varepsilon};\mathbb{R}^n)} &\leq C\|\nabla\phi\|_{L^2(\widetilde{\widetilde{\Omega}}_{3\sqrt{n}\varepsilon};\mathbb{R}^n)}\|\chi\|_{L^2(Y)}, \end{aligned} \tag{2.22}$$

2.4.2 Some complements

In this subsection, we extend some results given above to functions belonging to $H^1_\rho(\Omega)$. These technical complements intervene in the proofs of the projection theorems and in the Theorem 6.1.

Lemma 2.5. *For $\phi \in H^1_\rho(\Omega)$ we have*

$$\forall i \in \{1,\dots,n\}, \qquad \begin{aligned} &||\rho\big(\mathcal{M}_\varepsilon(\phi) - \phi\big)||_{L^2(\Omega)} \leq C\varepsilon||\phi||_\rho, \\ &||\rho\big(\phi(\cdot + \varepsilon\mathbf{e}_i) - \phi\big)||_{L^2(\Omega)} \leq C\varepsilon||\phi||_\rho, \\ &||\rho\big(\mathcal{M}_\varepsilon(\phi)(\cdot + \varepsilon\mathbf{e}_i) - \mathcal{M}_\varepsilon(\phi)\big)||_{L^2(\Omega)} \leq C\varepsilon||\phi||_\rho. \end{aligned} \tag{2.23}$$

For $\phi \in L^2_{1/\rho}(\Omega)$ we have

$$||\mathcal{M}_\varepsilon(\phi) - \phi||_{(H^1_\rho(\Omega))'} \leq C\varepsilon||\phi/\rho||_{L^2(\Omega)}. \tag{2.24}$$

The constants do not depend on ε.

Proof. Step 1. We prove $(2.23)_1$.
Let ϕ be in $H^1_\rho(\Omega)$ and let $\varepsilon(\xi + Y)$ be a cell included in Ω.
Case 1: $\rho(\varepsilon\xi) \geq 2\sqrt{n}\varepsilon$.

In this case, observing that

$$1 \leq \frac{\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}}{\min_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}} \leq 3$$

and thanks to the Poincaré-Wirtinger's inequality we obtain

$$\begin{aligned}
\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx &\leq [\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}]^2 \int_{\varepsilon(\xi+Y)} |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \\
&\leq [\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}]^2 C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\nabla\phi(x)|^2 dx \\
&\leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\nabla\phi(x)|^2 dx.
\end{aligned}$$

Case 2: $\rho(\varepsilon\xi) \leq 2\sqrt{n}\varepsilon$.
In this case we have

$$\int_{\varepsilon(\xi+Y)} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\varepsilon(\xi+Y)} |\phi(x)|^2 dx.$$

The cases 1 and 2 lead to

$$\int_{\widehat{\Omega}_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widehat{\Omega}_\varepsilon} \big([\rho(x)]^2 |\nabla\phi(x)|^2 + |\phi(x)|^2\big) dx. \tag{2.25}$$

Since $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$ and due to Lemma 2.1 we get

$$\int_{\Lambda_\varepsilon} [\rho(x)]^2 |\mathcal{M}_\varepsilon(\phi)(x) - \phi(x)|^2 dx \leq C\varepsilon^2 \int_{\widetilde{\Omega}_{c_0\sqrt{n}\varepsilon}} |\phi(x)|^2 dx$$

which in turn with (2.25) gives $(2.23)_1$. Proceeding in the same way we obtain $(2.23)_2$ and $(2.23)_3$.
Step 2. We prove (2.24).
Let ϕ be in $L^2_{1/\rho}(\Omega)$ and $\psi \in H^1_\rho(\Omega)$. We have

$$\int_{\widehat{\Omega}_\varepsilon} \big(\mathcal{M}_\varepsilon(\phi) - \phi\big)\psi = \int_{\widehat{\Omega}_\varepsilon} \big(\mathcal{M}_\varepsilon(\psi) - \psi\big)\phi.$$

Consequently we obtain

$$\begin{aligned}
\Big|\int_\Omega \big(\mathcal{M}_\varepsilon(\phi) - \phi\big)\psi - \int_\Omega \big(\mathcal{M}_\varepsilon(\psi) - \psi\big)\phi\Big| &\leq \int_{\Lambda_\varepsilon} \big|\big(\mathcal{M}_\varepsilon(\phi) - \phi\big)\psi\big| + \int_{\Lambda_\varepsilon} \big|\big(\mathcal{M}_\varepsilon(\psi) - \psi\big)\phi\big| \\
&\leq C\big(||\phi||_{L^2(\Lambda_\varepsilon)} + ||\mathcal{M}_\varepsilon(\phi)||_{L^2(\Lambda_\varepsilon)}\big)||\psi||_{L^2(\Omega)}.
\end{aligned}$$

The inclusion $\Lambda_\varepsilon \subset \widetilde{\Omega}_{\sqrt{n}\varepsilon}$, the fact that $\phi \in L^2_{1/\rho}(\Omega)$ and the estimates $(2.3)_1$-$(2.23)_1$ lead to

$$\int_\Omega \big(\mathcal{M}_\varepsilon(\phi) - \phi\big)\psi \leq C\varepsilon||\phi/\rho||_{L^2(\Omega)}||\psi||_\rho.$$

Hence (2.24) is proved. $\square$

Lemma 2.6. *For $\phi \in H^1_\rho(\Omega)$ we have*

$$||\rho\big(\mathcal{Q}_\varepsilon(\phi) - \phi\big)||_{L^2(\Omega)} \le C\varepsilon||\phi||_\rho \tag{2.26}$$

For $\phi \in H^1_{1/\rho}(\Omega)$ and ϕ_ε given by Lemma 2.4 we have

$$\begin{aligned}
&||\mathcal{Q}_\varepsilon(\phi_\varepsilon)||_{1/\rho} \le C||\phi||_{1/\rho}, \qquad \big\|\big(\phi - \mathcal{Q}_\varepsilon(\phi_\varepsilon)\big)/\rho\big\|_{L^2(\Omega)} \le C\varepsilon||\phi||_{1/\rho},\\
&\forall \mathbf{i} = i_1\mathbf{e}_1 + \ldots + i_n\mathbf{e}_n, \qquad (i_1, \ldots, i_n) \in \{0,1\}^n\\
&\big\|\big(\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot + \varepsilon\mathbf{i}) - \mathcal{M}_\varepsilon(\phi_\varepsilon)\big)/\rho\big\|_{L^2(\Omega)} \le C\varepsilon||\phi||_{1/\rho}.
\end{aligned} \tag{2.27}$$

For $\phi \in L^2(\mathbb{R}^n)$ and $\chi \in L^2(Y)$

$$\big\|\big(\mathcal{M}_\varepsilon(\rho\phi) - \rho\mathcal{M}_\varepsilon(\phi)\big)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\mathbb{R}^n)} \le C\varepsilon\|\phi\|_{L^2(\mathbb{R}^n)}\|\chi\|_{L^2(Y)}. \tag{2.28}$$

For $\phi \in H^1_\rho(\Omega)$ and $\chi \in L^2(Y)$

$$\begin{aligned}
&\big\|\rho\big(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi)\big)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\Omega)} \le C\varepsilon\|\phi\|_\rho\|\chi\|_{L^2(Y)},\\
&\big\|\rho\nabla\mathcal{Q}_\varepsilon(\phi)\chi\Big(\Big\{\frac{\cdot}{\varepsilon}\Big\}\Big)\big\|_{L^2(\Omega)} \le C\|\phi\|_\rho\|\chi\|_{L^2(Y)}.
\end{aligned} \tag{2.29}$$

The constants do not depend on ε.

Proof. Step 1. Let ϕ be in $H^1_\rho(\Omega)$. We first prove

$$||\rho\big(\mathcal{Q}_\varepsilon(\phi) - \mathcal{M}_\varepsilon(\phi)\big)||_{L^2(\Omega)} \le C\varepsilon||\phi||_\rho. \tag{2.30}$$

To do that, we proceed as in the proof of $(2.23)_1$. Let $\varepsilon(\xi + Y)$ be a cell included in Ω.
Case 1: $\rho(\varepsilon\xi) \ge 3\sqrt{n}\varepsilon$.
In this case we have

$$1 \le \frac{\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}}{\min_{z\in\varepsilon(\xi+2Y)}\{\rho(z)\}} \le 4 \qquad \text{and} \qquad 1 \le \frac{\max_{z\in\varepsilon(\xi+2Y)}\{\rho(z)\}}{\min_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}} \le \frac{5}{2}.$$

By definition of $\mathcal{Q}_\varepsilon(\phi)$ we deduce that

$$\begin{aligned}
\int_{\varepsilon(\xi+Y)} [\rho(x)]^2|\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2dx &\le [\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}]^2\int_{\varepsilon(\xi+Y)}|\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2dx\\
&\le [\max_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}]^2C\varepsilon^2\int_{\varepsilon(\xi+2Y)}|\nabla\phi(x)|^2dx\\
&\le C\varepsilon^2\int_{\varepsilon(\xi+2Y)}[\rho(x)]^2|\nabla\phi(x)|^2dx.
\end{aligned}$$

Case 2: $\rho(\varepsilon\xi) \le 3\sqrt{n}\varepsilon$. Then again by definition of $\mathcal{Q}_\varepsilon(\phi)$ we get

$$\int_{\varepsilon(\xi+Y)}[\rho(x)]^2|\mathcal{Q}_\varepsilon(\phi)(x) - \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi)|^2dx \le C\varepsilon^2\int_{\varepsilon(\xi+2Y)}|\phi(x)|^2dx.$$

As a consequence of both cases we get

$$\int_{\widehat{\Omega}_\varepsilon}[\rho(x)]^2|\mathcal{Q}_\varepsilon(\phi)(x)-\mathcal{M}_\varepsilon(\phi)(x)|^2dx\le C\varepsilon^2\int_\Omega\big([\rho(x)]^2|\nabla\phi(x)|^2+|\phi(x)|^2\big)dx. \quad (2.31)$$

Furthermore we have

$$\int_{\Lambda_\varepsilon}[\rho(x)]^2|\mathcal{Q}_\varepsilon(\phi)(x)|^2dx\le C\varepsilon^2\int_{\Lambda_\varepsilon}|\mathcal{Q}_\varepsilon(\phi)(x)|^2dx\le C\varepsilon^2\int_\Omega|\phi(x)|^2dx$$

which with (2.31) lead to (2.30). Then as a consequence of $(2.23)_1$ and (2.30) we get (2.26).

Step 2. We prove $(2.27)_1$.

Let ϕ be in $H^1_{1/\rho}(\Omega)$ and ϕ_ε given by Lemma 2.4. Due to the fact that $\phi_\varepsilon(x)=0$ for a.e. $x\in\mathbb{R}^n\setminus\overline{\widetilde{\Omega}}_{6\sqrt{n}\varepsilon}$, hence $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)=0$ for every $x\in\Omega$ such that $\rho(x)\le 4\sqrt{n}\varepsilon$. Again we take a cell $\varepsilon(\xi+Y)$ included in Ω such that $\rho(\varepsilon\xi)\ge 3\sqrt{n}\varepsilon$. The values taken by $\mathcal{Q}_\varepsilon(\phi_\varepsilon)$ in the cell $\varepsilon(\xi+Y)$ depend only on the values of ϕ_ε in $\varepsilon(\xi+2Y)$. Then we have

$$\begin{aligned}&\int_{\varepsilon(\xi+Y)}\frac{1}{[\rho(x)]^2}|\nabla\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2dx\le\frac{C}{[\min_{x\in\varepsilon(\xi+Y)}\{\rho(x)\}]^2}\int_{\varepsilon(\xi+2Y)}|\nabla\phi_\varepsilon(x)|^2dx\\&\le C\frac{[\max_{x\in\varepsilon(\xi+2Y)}\{\rho(x)\}]^2}{[\min_{x\in\varepsilon(\xi+Y)}\{\rho(x)\}]^2}\int_{\varepsilon(\xi+2Y)}\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx\le C\int_{\varepsilon(\xi+2Y)}\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx.\end{aligned}$$

Adding all these inequalities gives

$$\int_{\widetilde{\Omega}_{4\sqrt{n}\varepsilon}}\frac{1}{[\rho(x)]^2}|\nabla\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)|^2dx\le C\int_\Omega\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx$$

Since $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)=0$ for every $x\in\Omega$ such that $\rho(x)\le 4\sqrt{n}\varepsilon$, we get $||\mathcal{Q}_\varepsilon(\phi_\varepsilon)||_{1/\rho}\le C||\phi_\varepsilon||_{1/\rho}$. We conclude using $(2.10)_2$.

Step 3. Now we prove $(2.27)_2$. Again we consider a cell $\varepsilon(\xi+Y)$ included in Ω such that $\rho(\varepsilon\xi)\ge 3\sqrt{n}\varepsilon$. We have

$$\begin{aligned}&\int_{\varepsilon(\xi+Y)}\frac{1}{[\rho(x)]^2}|\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)-\phi_\varepsilon(x)|^2dx\le\frac{C}{[\min_{x\in\varepsilon(\xi+Y)}\{\rho(x)\}]^2}\int_{\varepsilon(\xi+Y)}|\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)-\phi_\varepsilon(x)|^2dx\\&\le\frac{C}{[\min_{x\in\varepsilon(\xi+Y)}\{\rho(x)\}]^2}\sum_{\mathbf{i}\in\{0,1\}^n}\int_{\varepsilon(\xi+\mathbf{i}+Y)}|\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi+\varepsilon\mathbf{i})-\phi_\varepsilon(x)|^2dx\\&\le C\varepsilon^2\frac{[\max_{z\in\varepsilon(\xi+2Y)}\{\rho(z)\}]^2}{[\min_{z\in\varepsilon(\xi+Y)}\{\rho(z)\}]^2}\int_{\varepsilon(\xi+2Y)}\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx\le C\varepsilon^2\int_{\varepsilon(\xi+2Y)}\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx.\end{aligned}$$

Hence we get

$$\int_{\widetilde{\Omega}_{4\sqrt{n}\varepsilon}}\frac{1}{[\rho(x)]^2}|\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)-\phi_\varepsilon(x)|^2dx\le C\varepsilon^2\int_\Omega\frac{1}{[\rho(x)]^2}|\nabla\phi_\varepsilon(x)|^2dx$$

The above estimate and the fact that $\mathcal{Q}_\varepsilon(\phi_\varepsilon)(x)-\phi_\varepsilon(x)=0$ for a.e. $x\in\Omega$ such that $\rho(x)\le 4\sqrt{n}\varepsilon$ yield $||(\phi_\varepsilon-\mathcal{Q}_\varepsilon(\phi_\varepsilon))/\rho||_{L^2(\Omega)}\le C\varepsilon||\phi_\varepsilon||_{1/\rho}$. We conclude using both estimates in (2.10).

Proceeding as in the Steps 2 and 3 we obtain $(2.27)_3$, (2.28) and (2.29). $\square$

3 Two new projection theorems

Theorem 3.1. *Let ϕ be in $H^1_{1/\rho}(\Omega)$. There exists $\widehat{\phi}_\varepsilon\in H^1_{per}(Y;L^2(\Omega))$ such that*

$$\begin{cases} ||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))}\le C\big\{||\phi||_{L^2(\Omega)}+\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\}\\ ||\mathcal{T}_\varepsilon(\phi)-\widehat{\phi}_\varepsilon||_{H^1(Y;(H^1_\rho(\Omega))')}\le C\varepsilon\big(||\phi/\rho||_{L^2(\Omega)}+\varepsilon||\phi||_{1/\rho}\big).\end{cases}\tag{3.1}$$

The constants depend only on n and $\partial\Omega$.

Proof. Here, we proceed as in the proof of Proposition 3.3 in [15]. We first reintroduce the open sets $\widehat{\Omega}_{\varepsilon,i}$ and the "double" unfolding operators $\mathcal{T}_{\varepsilon,i}$. We set

$$\widehat{\Omega}_{\varepsilon,i}=\widehat{\Omega}_\varepsilon\cap\big(\widehat{\Omega}_\varepsilon-\varepsilon\mathbf{e}_i\big),\qquad K_i=\text{interior}\big(\overline{Y}\cup(\mathbf{e}_i+\overline{Y})\big),\quad i\in\{1,\dots,n\}.$$

The unfolding operator $\mathcal{T}_{\varepsilon,i}$ from $L^2(\Omega)$ into $L^2(\Omega\times K_i)$ is defined by

$$\forall\psi\in L^2(\Omega),\qquad \mathcal{T}_{\varepsilon,i}(\psi)(x,y)=\begin{cases}\psi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y+\varepsilon y\Big) & \text{for } x\in\widehat{\Omega}_{\varepsilon,i}\text{ and for a.e. } y\in K_i,\\ 0 & \text{for } x\in\Omega\setminus\overline{\widehat{\Omega}}_{\varepsilon,i}\text{ and for a.e. } y\in K_i.\end{cases}$$

The restriction of $\mathcal{T}_{\varepsilon,i}(\psi)$ to $\widehat{\Omega}_{\varepsilon,i}\times Y$ is equal to $\mathcal{T}_\varepsilon(\psi)$.

Step 1. Let us first take $\phi\in L^2_{1/\rho}(\Omega)$. We set $\psi=\dfrac{1}{\rho}\phi$ and we evaluate the difference $\mathcal{T}_{\varepsilon,i}(\phi)(.,..+\mathbf{e}_i)-\mathcal{T}_{\varepsilon,i}(\phi)$ in $L^2(Y;(H^1_\rho(\Omega))')$. For any $\Psi\in H^1_\rho(\Omega)$ a change of variables gives for a. e. $y\in Y$

$$\begin{aligned}\int_\Omega\mathcal{T}_{\varepsilon,i}(\phi)(x,y+\mathbf{e}_i)\Psi(x)dx&=\int_{\widehat{\Omega}_{\varepsilon,i}}\mathcal{T}_\varepsilon(\phi)(x+\varepsilon\mathbf{e}_i,y)\Psi(x)dx\\&=\int_{\widehat{\Omega}_{\varepsilon,i}+\varepsilon\mathbf{e}_i}\mathcal{T}_\varepsilon(\phi)(x,y)\Psi(x-\varepsilon\mathbf{e}_i)dx.\end{aligned}$$

Then we obtain for a. e. $y\in Y$

$$\begin{aligned}&\Big|\int_\Omega\big\{\mathcal{T}_{\varepsilon,i}(\phi)(.,y+\mathbf{e}_i)-\mathcal{T}_{\varepsilon,i}(\phi)(.,y)\big\}\Psi-\int_{\widehat{\Omega}_{\varepsilon,i}}\mathcal{T}_\varepsilon(\psi)(.,y)\rho\big\{\Psi(.-\varepsilon\mathbf{e}_i)-\Psi\big\}\Big|\\ \le&\Big|\int_{\widehat{\Omega}_{\varepsilon,i}}\mathcal{T}_\varepsilon(\psi)(.,y)\big(\mathcal{T}_\varepsilon(\rho)-\rho\big)\big\{\Psi(.-\varepsilon\mathbf{e}_i)-\Psi\big\}\Big|+C||\mathcal{T}_\varepsilon(\phi)(.,y)||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}||\Psi||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}.\end{aligned}$$

Estimate $(2.23)_2$ leads to

$$||\rho\big(\Psi(. - \varepsilon\mathbf{e}_i) - \Psi\big)||_{L^2(\widehat{\Omega}_{\varepsilon,i})} \le C\varepsilon||\Psi||_\rho \qquad \forall i \in \{1,\dots,n\}.$$

We have

$$||\mathcal{T}_\varepsilon(\rho) - \rho||_{L^\infty(\Omega)} \le C\varepsilon. \tag{3.2}$$

The above inequalities imply

$$\begin{aligned}
&< \mathcal{T}_{\varepsilon,i}(\phi)(., y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(., y)\,,\ \Psi >_{(H^1_\rho(\Omega))', H^1_\rho(\Omega)}\\
&\quad = \int_\Omega \big\{\mathcal{T}_{\varepsilon,i}(\phi)(x, y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(x, y)\big\}\Psi(x)dx\\
&\quad \le C\varepsilon||\Psi||_\rho\|\mathcal{T}_\varepsilon(\psi)(., y)\|_{L^2(\Omega)} + C\varepsilon||\Psi||_{L^2(\Omega)}\|\mathcal{T}_\varepsilon(\psi)(., y)\|_{L^2(\Omega)}\\
&\quad + C||\mathcal{T}_\varepsilon(\phi)(., y)||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}||\Psi||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}.
\end{aligned}$$

Therefore, for a.e. $y \in Y$ we have

$$||\mathcal{T}_{\varepsilon,i}(\phi)(., y + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)(., y)||_{(H^1_\rho(\Omega))'} \le C\varepsilon\|\mathcal{T}_\varepsilon(\psi)(., y)\|_{L^2(\Omega)} + C||\mathcal{T}_\varepsilon(\phi)(., y)||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}$$

which leads to the following estimate of the difference between $\mathcal{T}_{\varepsilon,i}(\phi)_{|\Omega\times Y}$ and one of its translated :

$$\begin{aligned}
||\mathcal{T}_{\varepsilon,i}(\phi)(., .. + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)||_{L^2(Y;(H^1_\rho(\Omega))')} &\le C\varepsilon||\phi/\rho||_{L^2(\Omega)} + C||\phi||_{L^2(\widetilde{\Omega}_{2\sqrt{n}\varepsilon})}\\
&\le C\varepsilon||\phi/\rho||_{L^2(\Omega)}.
\end{aligned} \tag{3.3}$$

The constant depends only on the boundary of Ω.

Step 2. Let $\phi \in H^1_{1/\rho}(\Omega)$. The above estimate (3.3) applied to ϕ and its partial derivatives give

$$\begin{aligned}
||\mathcal{T}_{\varepsilon,i}(\phi)(., .. + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)||_{L^2(Y;(H^1_\rho(\Omega))')} &\le C\varepsilon||\phi/\rho||_{L^2(\Omega)}\\
||\mathcal{T}_{\varepsilon,i}(\nabla\phi)(., .. + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\nabla\phi)||_{[L^2(Y;(H^1_\rho(\Omega))']^n)} &\le C\varepsilon||\phi||_{1/\rho}.
\end{aligned}$$

which in turn lead to (we recall that $\nabla_y\big(\mathcal{T}_{\varepsilon,i}(\phi)\big) = \varepsilon\mathcal{T}_{\varepsilon,i}(\nabla\phi)$).

$$||\mathcal{T}_{\varepsilon,i}(\phi)(., .. + \mathbf{e}_i) - \mathcal{T}_{\varepsilon,i}(\phi)||_{H^1(Y;(H^1_\rho(\Omega))')} \le C\varepsilon\big(||\phi/\rho||_{L^2(\Omega)} + \varepsilon||\phi||_{1/\rho}\big).$$

From these inequalities for $i \in \{1,\dots,n\}$ we deduce the estimate of the difference of the traces of the function $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(., y)$ on the faces $Y_i \doteq \{y \in \overline{Y} \mid y_i = 0\}$ and $\mathbf{e}_i + Y_i$

$$||\mathcal{T}_\varepsilon(\phi)(., .. + \mathbf{e}_i) - \mathcal{T}_\varepsilon(\phi)||_{H^{1/2}(Y_i;(H^1_\rho(\Omega))')} \le C\varepsilon\big(||\phi/\rho||_{L^2(\Omega)} + \varepsilon||\phi||_{1/\rho}\big). \tag{3.4}$$

These estimates ($i \in \{1,\dots,n\}$) give a measure of the periodic defect of the function $y \longrightarrow \mathcal{T}_\varepsilon(\phi)(., y)$ (see [15]).

Then we decompose $\mathcal{T}_\varepsilon(\phi)$ into the sum of an element belonging to $H^1_{per}(Y;L^2(\Omega))$ and one to $\big(H^1(Y;L^2(\Omega))\big)^\perp$ (the orthogonal of $H^1_{per}(Y;L^2(\Omega))$ in $H^1(Y;L^2(\Omega))$, see [15])

$$\mathcal{T}_\varepsilon(\phi)=\widehat{\phi}_\varepsilon+\overline{\phi}_\varepsilon,\qquad \widehat{\phi}_\varepsilon\in H^1_{per}(Y;L^2(\Omega)),\qquad \overline{\phi}_\varepsilon\in\big(H^1(Y;L^2(\Omega))\big)^\perp. \tag{3.5}$$

The function $y\longrightarrow\mathcal{T}_\varepsilon(\phi)(.,y)$ takes its values in a finite dimensional space,

$$\overline{\phi}_\varepsilon(.,..)=\sum_{\xi\in\Xi_\varepsilon}\overline{\phi}_{\varepsilon,\xi}(..)\chi_{\varepsilon,\xi}(.)$$

where $\chi_{\varepsilon,\xi}(.)$ is the characteristic function of the cell $\varepsilon(\xi+Y)$ and where $\overline{\phi}_{\varepsilon,\xi}(..)\in\big(H^1(Y)\big)^\perp$ (the orthogonal of $H^1_{per}(Y)$ in $H^1(Y)$, see [15]). The decomposition (3.5) is the same in $H^1(Y;(H^1_\rho(\Omega))')$ and we have

$$||\widehat{\phi}_\varepsilon||^2_{H^1(Y;L^2(\Omega))}+||\overline{\phi}_\varepsilon||^2_{H^1(Y;L^2(\Omega))}=||\mathcal{T}_\varepsilon(\phi)||^2_{H^1(Y;L^2(\Omega))}\le C\big\{||\phi||_{L^2(\Omega)}+\varepsilon||\nabla\phi||_{[L^2(\Omega)]^n}\big\}^2.$$

It gives the first inequality in (3.1) and the estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y;L^2(\Omega))$. From Theorem 2.2 in [15] and (3.4) we obtain a finer estimate of $\overline{\phi}_\varepsilon$ in $H^1(Y;(H^1_\rho(\Omega))')$

$$||\overline{\phi}_\varepsilon||_{H^1(Y;(H^1_\rho(\Omega))')}\le C\varepsilon\big(||\phi/\rho||_{L^2(\Omega)}+\varepsilon||\phi||_{1/\rho}\big).$$

It is the second inequality in (3.1). $\square$

Theorem 3.2. *For $\phi\in H^1_{1/\rho}(\Omega)$, there exists $\widehat{\phi}_\varepsilon\in H^1_{per}(Y;L^2(\Omega))$ such that*

$$\begin{aligned}&||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))}\le C||\nabla\phi||_{[L^2(\Omega)]^n},\\ &||\mathcal{T}_\varepsilon(\nabla\phi)-\nabla\phi-\nabla_y\widehat{\phi}_\varepsilon||_{[L^2(Y;(H^1_\rho(\Omega))')]^n}\le C\varepsilon||\phi||_{1/\rho}.\end{aligned}\tag{3.6}$$

The constants depend only on $\partial\Omega$.

Proof. Let ϕ be in $H^1_{1/\rho}(\Omega)$ and $\psi=\phi/\rho\in H^1_0(\Omega)$. The function ϕ is extended by 0 outside of Ω. We decompose ϕ as

$$\phi=\Phi+\varepsilon\underline{\phi},\quad\text{where }\ \Phi=\mathcal{Q}_\varepsilon(\phi_\varepsilon)\quad\text{and}\quad\underline{\phi}=\frac{1}{\varepsilon}\Big(\phi-\mathcal{Q}_\varepsilon(\phi_\varepsilon)\Big)$$

where ϕ_ε is given by Lemma 2.4. We have Φ and $\underline{\phi}\in H^1_0(\Omega)$ and due to (2.27) we get the following estimates:

$$||\Phi||_{1/\rho}+\varepsilon||\underline{\phi}||_{1/\rho}+||\underline{\phi}/\rho||_{L^2(\Omega)}\le C||\phi||_{1/\rho}. \tag{3.7}$$

The projection Theorem 3.1 applied to $\underline{\phi}\in H^1_{1/\rho}(\Omega)$ gives an element $\widehat{\phi}_\varepsilon$ in $H^1_{per}(Y;L^2(\Omega))$ such that

$$\begin{aligned}&||\widehat{\phi}_\varepsilon||_{H^1(Y;L^2(\Omega))}\le C||\phi||_{1/\rho},\\ &||\mathcal{T}_\varepsilon(\underline{\phi})-\widehat{\phi}_\varepsilon||_{H^1(Y;(H^1_\rho(\Omega))')}\le C\varepsilon||\phi||_{1/\rho}.\end{aligned}\tag{3.8}$$

Now we evaluate $||\mathcal{T}_\varepsilon(\nabla\Phi)-\nabla\Phi||_{[L^2(Y;(H^1_\rho(\Omega))')]^n}$.

From (2.24), $(2.27)_1$ and (3.7) we get

$$\|\nabla\Phi-\mathcal{M}_\varepsilon(\nabla\Phi)\|_{(H^1_\rho(\Omega;\mathbb{R}^n))'}\le C\varepsilon\|\phi\|_{1/\rho}. \tag{3.9}$$

We set

$$H^{(1)}(z)=\begin{cases}\big(1-|z_2|\big)\big(1-|z_3|\big)\dots\big(1-|z_n|\big) & \text{if} \quad z=(z_1,z_2,\dots,z_n)\in[-1,1]^n,\\ 0 & \text{if} \quad z\in\mathbb{R}^n\setminus[-1,1]^n.\end{cases}$$

$$\mathbf{I}=\Big\{\mathbf{i}\mid \mathbf{i}=i_2\mathbf{e}_2+\dots+i_n\mathbf{e}_n,\quad (i_2,\dots,i_n)\in\{0,1\}^{n-1}\Big\}$$

For $\xi\in\mathbb{Z}^n$ and for every $(x,y)\in\varepsilon(\xi+Y)\times Y$ we have

$$\begin{aligned}\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x,y)&=\sum_{\mathbf{i}\in\mathbf{I}}\frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{e}_1+\mathbf{i})\big)-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{i})\big)}{\varepsilon}H^{(1)}(y-\mathbf{i})\\ \mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(\varepsilon\xi)&=\frac{1}{2^{n-1}}\sum_{\mathbf{i}\in\mathbf{I}}\frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{e}_1+\mathbf{i})\big)-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{i})\big)}{\varepsilon}.\end{aligned}$$

Now, let us take $\psi\in H^1_\rho(\Omega)$. We recall that $\phi_\varepsilon(x)=0$ for a.e. $x\in\mathbb{R}^n\setminus\overline{\widehat{\Omega}}_{6\sqrt{n}\varepsilon}$, hence $\Phi(x)=0$ for $x\in\mathbb{R}^n\setminus\overline{\widehat{\Omega}}_{3\sqrt{n}\varepsilon}$; as a first consequence $\mathcal{M}_\varepsilon\Big(\dfrac{\partial\Phi}{\partial x_1}\Big)=0$ in Λ_ε.

For $y\in Y$ we have

$$\begin{aligned}<\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(.,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big),\psi>_{(H^1_\rho(\Omega))',H^1_\rho(\Omega)}&=\int_\Omega\Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x)\Big\}\psi(x)dx\\ &=\int_{\widehat{\Omega}_\varepsilon}\Big\{\mathcal{T}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x)\Big\}\mathcal{M}_\varepsilon(\psi)(x)dx.\end{aligned}$$

Besides we have

$$\begin{aligned}\int_{\widehat{\Omega}_\varepsilon}\mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(x)\mathcal{M}_\varepsilon(\psi)(x)dx&=\varepsilon^n\sum_{\xi\in\mathbb{Z}^n}\mathcal{M}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_1}\Big)(\varepsilon\xi)\mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)\\ &=\frac{\varepsilon^n}{2^{n-1}}\sum_{\xi\in\mathbb{Z}^n}\sum_{\mathbf{i}\in\mathbf{I}}\frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{e}_1+\mathbf{i})\big)-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{i})\big)}{\varepsilon}\mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)\\ &=\frac{\varepsilon^n}{2^{n-1}}\sum_{\xi\in\mathbb{Z}^n}\sum_{\mathbf{i}\in\mathbf{I}}\frac{\mathcal{M}_\varepsilon(\psi)\big(\varepsilon(\xi-\mathbf{e}_1)\big)-\mathcal{M}_\varepsilon(\psi)\big(\varepsilon\xi\big)}{\varepsilon}\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i}))\end{aligned}$$

and

$$\begin{aligned}
&\int_{\widehat{\Omega}_\varepsilon} \mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)(x,y)\mathcal{M}_\varepsilon(\psi)(x)dx\\
=&\varepsilon^n \sum_{\xi\in\mathbb{Z}^n}\sum_{\mathbf{i}\in\mathbf{I}} \Big[\frac{\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{e}_1+\mathbf{i})\big)-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big(\varepsilon(\xi+\mathbf{i})\big)}{\varepsilon}\Big]H^{(1)}(y-\mathbf{i})\mathcal{M}_\varepsilon(\psi)(\varepsilon\xi)\\
=&\varepsilon^n \sum_{\xi\in\mathbb{Z}^n}\sum_{\mathbf{i}\in\mathbf{I}} \frac{\mathcal{M}_\varepsilon(\psi)\big(\varepsilon(\xi-\mathbf{e}_1)\big)-\mathcal{M}_\varepsilon(\psi)\big(\varepsilon\xi\big)}{\varepsilon}H^{(1)}(y-\mathbf{i})\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i}))
\end{aligned}$$

Due to the fact that $\phi_\varepsilon(x)=0$ for a.e. $x\in\mathbb{R}^n\setminus\overline{\widehat{\Omega}}_{6\sqrt{n}\varepsilon}$, in the above summations we only take the ξ's belonging to Ξ_ε and satisfying $\rho(\varepsilon\xi)\geq 3\sqrt{n}\varepsilon$. Hence

$$\begin{aligned}
&< \mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)(.,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big),\psi>_{(H^1_\rho(\Omega))',H^1_\rho(\Omega)}\\
=&\varepsilon^n \sum_{\xi\in\mathbb{Z}^n} \frac{\mathcal{M}_\varepsilon(\psi)\big(\varepsilon(\xi-\mathbf{e}_1)\big)-\mathcal{M}_\varepsilon(\psi)\big(\varepsilon\xi\big)}{\varepsilon}\sum_{\mathbf{i}\in\mathbf{I}}\big[H^{(1)}(y-\mathbf{i})-\frac{1}{2^{n-1}}\big]\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i})).
\end{aligned}$$

Thanks to the identity relation $\displaystyle\sum_{\mathbf{i}\in\mathbf{I}}\big[H^{(1)}(y-\mathbf{i})-\frac{1}{2^{n-1}}\big]=0$ we obtain that

$$\Big|\sum_{\mathbf{i}\in\mathbf{I}}\big[H^{(1)}(y-\mathbf{i})-\frac{1}{2^{n-1}}\big]\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i}))\Big|\leq\sum_{\mathbf{i}\in\mathbf{I}}\big|\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i}))-\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi)\big|.$$

Taking into account the last equality and inequality above we deduce that

$$\begin{aligned}
&< \mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)(.,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big),\psi>_{(H^1_\rho(\Omega))',H^1_\rho(\Omega)}\\
=&\varepsilon^n \sum_{\xi\in\mathbb{Z}^n}\sum_{\mathbf{i}\in\mathbf{I}}\Big|\frac{\mathcal{M}_\varepsilon(\psi)\big(\varepsilon(\xi-\mathbf{e}_1)\big)-\mathcal{M}_\varepsilon(\psi)\big(\varepsilon\xi\big)}{\varepsilon}\Big|\big|\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon(\xi+\mathbf{i}))-\mathcal{M}_\varepsilon(\phi_\varepsilon)(\varepsilon\xi)\big|\\
=&\frac{1}{\varepsilon}\sum_{\mathbf{i}\in\mathbf{I}}\int_\Omega\big|\mathcal{M}_\varepsilon(\psi)(\cdot-\varepsilon\mathbf{e}_1)-\mathcal{M}_\varepsilon(\psi)\big|\,\big|\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot+\varepsilon\mathbf{i})-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big|\\
\leq&\frac{C}{\varepsilon}\sum_{\mathbf{i}\in\mathbf{I}}\big\|\rho\big(\mathcal{M}_\varepsilon(\psi)(\cdot-\varepsilon\mathbf{e}_1)-\mathcal{M}_\varepsilon(\psi)\big)\big\|_{L^2(\Omega)}\Big\|\frac{1}{\rho}\big(\mathcal{M}_\varepsilon(\phi_\varepsilon)(\cdot+\varepsilon\mathbf{i})-\mathcal{M}_\varepsilon(\phi_\varepsilon)\big)\Big\|_{L^2(\Omega)}.
\end{aligned}$$

Due to $(2.23)_3$ and $(2.27)_3$ we finally get

$$< \mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)(.,y)-\mathcal{M}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big),\psi>_{(H^1_\rho(\Omega))',H^1_\rho(\Omega)}\leq C\varepsilon||\phi_\varepsilon||_{1/\rho}||\psi||_\rho.$$

It leads to

$$\Big\|\mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)-\mathcal{M}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big)\Big\|_{L^\infty(Y;(H^1_\rho(\Omega))')}\leq C\varepsilon||\phi_\varepsilon||_{1/\rho}. \tag{3.10}$$

Besides we have

$$\int_\Omega \frac{\partial \underline{\phi}}{\partial x_1}(x)\psi(x)dx = -\int_\Omega \underline{\phi}(x)\frac{\partial \psi}{\partial x_1}(x)dx \le C||\underline{\phi}/\rho||_{L^2(\Omega)}||\psi||_\rho \le C||\phi||_{1/\rho}||\psi||_\rho.$$

Hence $\Big\|\varepsilon\dfrac{\partial \underline{\phi}}{\partial x_1}\Big\|_{(H^1_\rho(\Omega;\mathbb{R}^n))'} \le C\varepsilon||\phi||_{1/\rho}$. This last estimate with $(2.10)_2$, (3.9) and (3.10) yield

$$\Big\|\mathcal{T}_\varepsilon\Big(\frac{\partial \Phi}{\partial x_1}\Big) - \frac{\partial \phi}{\partial x_1}\Big\|_{L^\infty(Y;(H^1_\rho(\Omega))')} \le C\varepsilon||\phi_\varepsilon||_{1/\rho}.$$

In the same way we prove the estimates for the partial derivatives of Φ with respect to x_i, $i \in \{2,\dots,n\}$. Hence we get $\|\mathcal{T}_\varepsilon(\nabla\Phi) - \nabla\phi\|_{[L^\infty(Y;(H^1_\rho(\Omega))')]^n} \le C\varepsilon||\phi_\varepsilon||_{1/\rho}$. Then thanks to (3.8) the second estimate in (3.6) is proved. $\square$

4 Reminds about the classical periodic homogenization problem

We consider the homogenization problem

$$\phi^\varepsilon \in H^1_0(\Omega), \qquad \int_\Omega A_\varepsilon(x)\nabla\phi^\varepsilon(x)\nabla\psi(x)dx = \int_\Omega f(x)\psi(x)dx, \qquad \forall \psi \in H^1_0(\Omega), \quad (4.1)$$

where

- $A_\varepsilon(x) = A\Big(\Big\{\dfrac{x}{\varepsilon}\Big\}\Big)$ for a.e. $x \in \Omega$, where A is a square matrix belonging to $L^\infty(Y;\mathbb{R}^{n\times n})$ and satisfying the condition of uniform ellipticity $c|\xi|^2 \le A(y)\xi\cdot\xi$ for a.e. $y \in Y$, with c a strictly positive constant,
- $f \in L^2(\Omega)$.

We showed in [10] that

$$\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon) \longrightarrow \nabla\Phi + \nabla_y\widehat{\phi} \quad \text{strongly in} \quad L^2(\Omega\times Y;\mathbb{R}^n)$$

where $(\Phi,\widehat{\phi}) \in H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y))$ is the solution of the problem of unfolding homogenization

$$\begin{aligned}
&\forall(\Psi,\widehat{\psi}) \in H^1_0(\Omega)\times L^2(\Omega;H^1_{per}(Y))\\
&\int_\Omega\int_Y A(y)\big\{\nabla\Phi(x) + \nabla_y\widehat{\phi}(x,y)\big\}\big\{\nabla\Psi(x) + \nabla_y\widehat{\psi}(x,y)\big\}dxdy = \int_\Omega f(x)\Psi(x)dx.
\end{aligned}$$

The correctors χ_i, $i \in \{1,\dots,n\}$, are the solutions of the variational problems

$$\begin{aligned}
&\chi_i \in H^1_{per}(Y), \qquad \int_Y \chi_i = 0,\\
&\int_Y A(y)\nabla_y\big(\chi_i(y) + y_i\big)\nabla_y\psi(y)dy = 0, \qquad \forall\psi \in H^1_{per}(Y).
\end{aligned} \tag{4.2}$$

They allow to express $\widehat{\phi}$ in terms of the partial derivatives of Φ

$$\widehat{\phi} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} \chi_i \tag{4.3}$$

and to give the homogenized problem satisfied by Φ

$$\Phi \in H_0^1(\Omega), \qquad \int_\Omega \mathcal{A}\nabla\Phi(x)\nabla\Psi(x)dx = \int_\Omega f(x)\Psi(x)dx, \qquad \forall \Psi \in H_0^1(\Omega) \tag{4.4}$$

where (see [10])

$$\mathcal{A}_{ij} = \sum_{k,l=1}^{n} \int_Y a_{kl}(y) \frac{\partial(y_j + \chi_j(y))}{\partial y_l} \frac{\partial(y_i + \chi_i(y))}{\partial y_k} dy. \tag{4.5}$$

5 An operator from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$

From now on, Ω is a bounded domain with a $\mathcal{C}^{1,1}$ boundary or an open bounded convex set.

In this section we first introduce a lifting operator $\mathbf{T}$ (defined by (5.1)) from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$. This operator and the estimate (5.2) are in fact sufficient to obtain the error estimates with a non-homogeneous Dirichlet condition (Theorem 6.3); one of the aim of this paper. Then we extend this operator. The extension of $\mathbf{T}$ from $H^{-1/2}(\partial\Omega)$ into $H^1_\rho(\Omega)$ is essential in order to get a sharper estimate (6.3) than $(6.2)_1$. In Theorem 7.1 we give an application based on (6.3), in this theorem we investigate a first case of strongly oscillating boundary data.

Let g be in $H^{1/2}(\partial\Omega)$, there exists one $\phi_g \in H^1(\Omega)$ such that

$$\operatorname{div}(\mathcal{A}\nabla\phi_g) = 0 \quad \text{in} \quad \Omega, \qquad \phi_g = g \quad \text{on} \quad \partial\Omega \tag{5.1}$$

where $\mathcal{A}$ is the matrix given by (4.5). We have

$$||\phi_g||_{H^1(\Omega)} \leq C||g||_{H^{1/2}(\partial\Omega)}. \tag{5.2}$$

We denote by $\mathbf{T}$ the operator from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$ which associates to $g \in H^{1/2}(\partial\Omega)$ the function $\phi_g \in H^1(\Omega)$.

Now, let (ψ, Ψ) be a couple in $[\mathcal{C}^\infty(\overline{\Omega})]^2$, integrating by parts over Ω gives

$$\int_\Omega \mathcal{A}\nabla\psi(x)\nabla\Psi(x)dx = -\int_\Omega \psi(x)\operatorname{div}(\mathcal{A}^T\nabla\Psi)(x)dx + \int_{\partial\Omega} \psi(x)(\mathcal{A}^T\nabla\Psi)(x)dx \cdot \nu(x)d\sigma.$$

The space $\mathcal{C}^\infty(\overline{\Omega})$ being dense in $H^1(\Omega)$ and $H^2(\Omega)$, hence the above equality holds true for any $\psi \in H^1(\Omega)$ and any $\Psi \in H^2(\Omega)$. Hence, for $\Psi \in H_0^1(\Omega) \cap H^2(\Omega)$ and ϕ_g defined by (5.1) we get

$$\int_\Omega \phi_g(x)\operatorname{div}(\mathcal{A}^T\nabla\Psi)(x)dx = \int_{\partial\Omega} g(x)\,(\mathcal{A}^T\nabla\Psi)(x) \cdot \nu(x)d\sigma. \tag{5.3}$$

Under the assumption on Ω the function $\Psi(g)$ defined by

$$\Psi(g) \in H^1_0(\Omega), \qquad \mathrm{div}(\mathcal{A}^T \nabla \Psi(g)) = \phi_g \qquad \text{in} \quad \Omega$$

belongs to $H^1_0(\Omega) \cap H^2(\Omega)$ and satisfies

$$||\Psi(g)||_{H^2(\Omega)} \leq C||\phi_g||_{L^2(\Omega)}.$$

Taking $\Psi = \Psi(g)$ in the above equality (5.3) we obtain

$$\begin{aligned}\int_\Omega |\phi_g(x)|^2 dx = \int_{\partial\Omega} g(x) \left(\mathcal{A}^T \nabla \Psi(g)(x)\right) \cdot \nu(x) d\sigma &\leq ||g||_{H^{-1/2}(\partial\Omega)} ||(\mathcal{A}^T \nabla \Psi(g)) \cdot \nu||_{H^{1/2}(\partial\Omega)} \\ &\leq C||g||_{H^{-1/2}(\partial\Omega)} ||\Psi(g)||_{H^2(\Omega)}.\end{aligned}$$

This leads to

$$||\phi_g||_{L^2(\Omega)} \leq C||g||_{H^{-1/2}(\partial\Omega)}. \tag{5.4}$$

Due to (5.4), the operator $\mathbf{T}$ admits an extension (still denoted $\mathbf{T}$) from $H^{-1/2}(\partial\Omega)$ into $L^2(\Omega)$ and we have

$$\forall g \in H^{-1/2}(\partial\Omega), \qquad ||\mathbf{T}(g)||_{L^2(\Omega)} \leq C||g||_{H^{-1/2}(\partial\Omega)}.$$

For $g \in H^{-1/2}(\partial\Omega)$, we also denote $\phi_g = \mathbf{T}(g)$. This function is the "very weak" solution of the problem

$$\phi_g \in L^2(\Omega), \qquad \mathrm{div}(\mathcal{A}\nabla\phi_g) = 0 \qquad \text{in} \quad \Omega, \qquad \phi_g = g \qquad \text{on} \quad \partial\Omega$$

or the solution of the following:

$$\begin{aligned}&\phi_g \in L^2(\Omega), \\ &\int_\Omega \phi_g(x)\, \mathrm{div}(\mathcal{A}^T \nabla \psi(x)) dx = < g, (\mathcal{A}^T \nabla\psi) \cdot \nu >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \\ &\forall \psi \in H^1_0(\Omega) \cap H^2(\Omega).\end{aligned} \tag{5.5}$$

Lemma 5.1. *The operator* $\mathbf{T}$ *is a bicontinuous linear operator from* $H^{-1/2}(\partial\Omega)$ *onto*

$$\mathbf{H} = \Big\{ \phi \in L^2(\Omega) \ \mid \ div(\mathcal{A}\nabla\phi) = 0 \quad in \quad \Omega \Big\}.$$

There exists a constant $C \geq 1$ *such that*

$$\forall g \in H^{-1/2}(\partial\Omega), \qquad \frac{1}{C}||g||_{H^{-1/2}(\partial\Omega)} \leq ||\mathbf{T}(g)||_{L^2(\Omega)} \leq C||g||_{H^{-1/2}(\partial\Omega)}. \tag{5.6}$$

Proof. Let ϕ be in $\mathbf{H}$ we are going to prove that there exists an element $g \in H^{-1/2}(\partial\Omega)$ such that $\mathbf{T}(g) = \phi$. To do that, we consider a continuous linear lifting operator $\mathbf{R}$ from $H^{1/2}(\partial\Omega)$ into $H^1_0(\Omega) \cap H^2(\Omega)$ satisfying for any $h \in H^{1/2}(\partial\Omega)$

$$\begin{aligned}&\mathbf{R}(h) \in H^1_0(\Omega) \cap H^2(\Omega), \\ &\mathcal{A}^T \nabla \mathbf{R}(h)_{|\partial\Omega} \cdot \nu = h \qquad \text{on} \quad \partial\Omega, \\ &||\mathbf{R}(h)||_{H^2(\Omega)} \leq C||h||_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

The map $h \longmapsto \int_\Omega \phi \, \mathrm{div}(\mathcal{A}^T \nabla \mathbf{R}(h))$ is a continuous linear form defined over $H^{1/2}(\partial\Omega)$. Thus, there exists $g \in H^{-1/2}(\partial\Omega)$ such that

$$\int_\Omega \phi \, \mathrm{div}(\mathcal{A}^T \nabla \mathbf{R}(h)) =< g, h >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} . \tag{5.7}$$

Since $\phi \in \mathbf{H}$, we deduce that for any $\psi \in \mathcal{C}_0^\infty(\Omega)$ we have $\int_\Omega \phi \, \mathrm{div}(\mathcal{A}^T \nabla \psi) = 0$. Therefore, for any $\psi \in H_0^2(\Omega)$ we have $\int_\Omega \phi \, \mathrm{div}(\mathcal{A}^T \nabla \psi) = 0$. Taking into account (5.7) we get

$$\int_\Omega \phi \, \mathrm{div}(\mathcal{A}^T \nabla \psi) =< g, (\mathcal{A}^T \nabla \psi) \cdot \nu >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}, \qquad \forall \psi \in H_0^1(\Omega) \cap H^2(\Omega).$$

It yields $\phi = \phi_g$ and then (5.6). $\square$

Remark 5.2. *It is well known (see e.g. [18]) that every function $\phi \in \mathbf{H}$ also belongs to $H^1_\rho(\Omega)$ and verifies*

$$||\phi||_\rho \le C||\phi||_{L^2(\Omega)}. \tag{5.8}$$

6 Error estimates with a non-homogeneous Dirichlet condition

Theorem 6.1. *Let $\big(\phi^\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions belonging to $H^1(\Omega)$ such that*

$$div\big(A_\varepsilon \nabla \phi^\varepsilon\big) = 0 \qquad in \quad \Omega. \tag{6.1}$$

Setting $g_\varepsilon = \phi^\varepsilon_{|\partial\Omega}$ and $\phi_{g_\varepsilon} = \mathbf{T}(g_\varepsilon) \in H^1(\Omega)$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$ we have

$$\begin{aligned} &||\phi^\varepsilon||_{H^1(\Omega)} \le C||g_\varepsilon||_{H^{1/2}(\partial\Omega)}, \qquad\qquad ||\phi^\varepsilon - \phi_{g_\varepsilon}||_{L^2(\Omega)} \le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}, \\ &\Big\|\rho\Big(\nabla\phi^\varepsilon - \nabla\phi_{g_\varepsilon} - \sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial \phi_{g_\varepsilon}}{\partial x_i}\Big)\nabla_y \chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)} \le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}. \end{aligned} \tag{6.2}$$

Moreover we have

$$||\phi^\varepsilon||_\rho \le C\big(\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)} + ||g_\varepsilon||_{H^{-1/2}(\partial\Omega)}\big). \tag{6.3}$$

The χ_i's are the correctors introduced in Section 4 and $\mathbf{T}$ is the operator defined in Section 5.

Proof. Step 1. We prove the first estimate in (6.2). From Section 5 we get

$$||\phi_{g_\varepsilon}||_{H^1(\Omega)} \leq C||g_\varepsilon||_{H^{1/2}(\partial\Omega)} \qquad ||\phi_{g_\varepsilon}||_\rho \leq C||g_\varepsilon||_{H^{-1/2}(\partial\Omega)}. \tag{6.4}$$

We write (6.1) in the following weak form:

$$\begin{aligned} &\phi^\varepsilon = \check{\phi}_\varepsilon + \phi_{g_\varepsilon}, \quad \check{\phi}_\varepsilon \in H^1_0(\Omega) \\ &\int_\Omega A_\varepsilon \nabla \check{\phi}_\varepsilon \nabla v = -\int_\Omega A_\varepsilon \nabla \phi_{g_\varepsilon} \nabla v \qquad \forall v \in H^1_0(\Omega). \end{aligned} \tag{6.5}$$

The solution $\check{\phi}_\varepsilon$ of the above variational problem satisfies

$$||\check{\phi}_\varepsilon||_{H^1(\Omega)} \leq C||\nabla \phi_{g_\varepsilon}||_{L^2(\Omega;\mathbb{R}^n)}.$$

Hence, from $(6.4)_1$ and the above estimate we get the first inequality in (6.2).

Step 2. We prove the second estimate in (6.2).

For every test function $v \in H^1_0(\Omega)$ we have

$$\int_\Omega A_\varepsilon \nabla \phi^\varepsilon \nabla v = 0. \tag{6.6}$$

Now, in order to obtain the L^2 error estimate we proceed as in the proof of the Theorem 3.2 in [16]. We first recall that for any $\phi \in H^1(\Omega)$ we have (see Lemma 2.3) for every $\varepsilon \leq \varepsilon_0 \doteq \gamma_0/3\sqrt{n}$

$$||\phi||_{L^2(\widetilde{\Omega}_{3c_0\sqrt{n}\varepsilon})} \leq C\varepsilon^{1/2}||\phi||_{H^1(\Omega)}.$$

Let U be a test function belonging to $H^1_0(\Omega) \cap H^2(\Omega)$. The above estimate yields

$$||\nabla U||_{L^2(\widetilde{\Omega}_{3c_0\sqrt{n}\varepsilon};\mathbb{R}^n)} \leq C\varepsilon^{1/2}||U||_{H^2(\Omega)} \tag{6.7}$$

which in turn with (2.12)-(2.13)-$(2.14)_1$ and $(6.2)_1$-(6.6) lead to

$$\Big|\int_{\Omega\times Y} A(y)\mathcal{T}_\varepsilon(\nabla\phi^\varepsilon)(x,y)\nabla U(x)dxdy\Big| \leq C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}. \tag{6.8}$$

The Theorem 2.3 in [16] gives an element $\widehat{\phi}_\varepsilon \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\begin{aligned} ||\mathcal{T}(\nabla\phi^\varepsilon) - \nabla\phi^\varepsilon - \nabla_y\widehat{\phi}_\varepsilon||_{[L^2(Y;(H^1(\Omega))')]^n} &\leq C\varepsilon^{1/2}||\nabla\phi^\varepsilon||_{L^2(\Omega;\mathbb{R}^n)} \\ &\leq C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}. \end{aligned} \tag{6.9}$$

The above inequalities (6.8) and (6.9) yield

$$\Big|\int_{\Omega\times Y} A\big(\nabla\phi^\varepsilon + \nabla_y\widehat{\phi}_\varepsilon\big)\nabla U\Big| \leq C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}. \tag{6.10}$$

We set

$$\forall x \in \mathbb{R}^n, \qquad \rho_\varepsilon(x) = \inf\Big\{1, \frac{\rho(x)}{\varepsilon}\Big\}.$$

Now, we take $\overline{\chi} \in H^1_{per}(Y)$ and we consider the test function $u_\varepsilon \in H^1_0(\Omega)$ defined for a.e. $x \in \Omega$ by

$$u_\varepsilon(x) = \varepsilon \rho_\varepsilon(x) \mathcal{Q}_\varepsilon\Big(\frac{\partial U}{\partial x_i}\Big)(x) \overline{\chi}\Big(\frac{x}{\varepsilon}\Big).$$

Due to $(2.21)_2$ and (6.7) we get

$$\Big\| \mathcal{Q}_\varepsilon\Big(\frac{\partial U}{\partial x_i}\Big) \nabla_y \overline{\chi}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^2(\widetilde{\Omega}_{\sqrt{n}\varepsilon};\mathbb{R}^n)} \le C\varepsilon^{1/2} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)} \tag{6.11}$$

Then by a straightforward calculation and thanks to $(2.21)_2$-$(2.22)_2$ and (6.7)-(6.11) we obtain

$$\Big\| \nabla u_\varepsilon - \mathcal{Q}_\varepsilon\Big(\frac{\partial U}{\partial x_i}\Big) \nabla_y \overline{\chi}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)} \le C\varepsilon^{1/2} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)}$$

which in turn with again (6.11) give

$$\|\nabla u_\varepsilon\|_{L^2(\widetilde{\Omega}_{\sqrt{n}\varepsilon};\mathbb{R}^n)} \le C\varepsilon^{1/2} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)} \tag{6.12}$$

and then with $(2.22)_1$ they lead to

$$\Big\| \nabla u_\varepsilon - \mathcal{M}_\varepsilon\Big(\frac{\partial U}{\partial x_i}\Big) \nabla_y \overline{\chi}\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)} \le C\varepsilon^{1/2} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)}.$$

In (6.6) we replace ∇u_ε with $\mathcal{M}_\varepsilon\Big(\dfrac{\partial U}{\partial x_i}\Big) \nabla_y \overline{\chi}\Big(\dfrac{\cdot}{\varepsilon}\Big)$; we continue using (2.12)-(2.13) and $(6.2)_1$-(6.12) to obtain

$$\Big| \int_{\Omega\times Y} A(y) \mathcal{T}_\varepsilon(\nabla \phi^\varepsilon)(x,y) \mathcal{M}_\varepsilon\Big(\frac{\partial U}{\partial x_i}\Big)(x) \nabla_y \overline{\chi}(y) dx dy\Big| \le C\varepsilon^{1/2} ||g_\varepsilon||_{H^{1/2}(\partial\Omega)} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)}$$

which with $(2.17)_2$ and then (6.9) give

$$\Big| \int_{\Omega\times Y} A(y) \big(\nabla \phi^\varepsilon(x) + \nabla_y \widehat{\phi}_\varepsilon(x,y)\big) \frac{\partial U}{\partial x_i}(x) \nabla_y \overline{\chi}(y) dx\, dy\Big| \le C\varepsilon^{1/2} ||g_\varepsilon||_{H^{1/2}(\partial\Omega)} ||U||_{H^2(\Omega)} ||\overline{\chi}||_{H^1(Y)}. \tag{6.13}$$

As in [16] we introduce the adjoint correctors $\overline{\chi}_i \in H^1_{per}(Y)$, $i \in \{1, \dots, n\}$, defined by

$$\int_Y A(y) \nabla_y \psi(y) \nabla_y (\overline{\chi}_i(y) + y_i) dy = 0 \qquad \forall \psi \in H^1_{per}(Y). \tag{6.14}$$

From (6.13) we get

$$\Big| \int_{\Omega\times Y} A\big(\nabla \phi^\varepsilon + \nabla_y \widehat{\phi}_\varepsilon\big) \nabla_y \Big(\sum_{i=1}^n \frac{\partial U}{\partial x_i} \overline{\chi}_i\Big)\Big| \le C\varepsilon^{1/2} ||g_\varepsilon||_{H^{1/2}(\partial\Omega)} ||U||_{H^2(\Omega)}$$

and from the definition (4.2) of the correctors χ_i we have

$$\int_{\Omega\times Y} A\Big(\nabla\phi^\varepsilon+\sum_{i=1}^n\frac{\partial\phi^\varepsilon}{\partial x_i}\nabla_y\chi_i\Big)\nabla_y\Big(\sum_{j=1}^n\frac{\partial U}{\partial x_j}\overline{\chi}_j\Big)=0.$$

Thus

$$\Big|\int_{\Omega\times Y} A\nabla_y\Big(\widehat{\phi}_\varepsilon-\sum_{i=1}^n\frac{\partial\phi^\varepsilon}{\partial x_i}\chi_i\Big)\nabla_y\Big(\sum_{j=1}^n\frac{\partial U}{\partial x_j}\overline{\chi}_j\Big)\Big|\le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}$$

and thanks to (6.14) we obtain

$$\Big|\int_{\Omega\times Y} A\nabla_y\Big(\widehat{\phi}_\varepsilon-\sum_{i=1}^n\frac{\partial\phi^\varepsilon}{\partial x_i}\chi_i\Big)\nabla U\Big|\le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}.$$

The above estimate, (6.10) and the expression (4.5) of the matrix $\mathcal{A}$ yield

$$\Big|\int_\Omega \mathcal{A}\nabla\phi^\varepsilon\nabla U\Big|\le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}.$$

Finally, since we have $\displaystyle\int_\Omega\mathcal{A}\nabla\phi_{g_\varepsilon}\nabla v=0$ for any $v\in H^1_0(\Omega)$, we deduce that

$$\forall U\in H^1_0(\Omega)\cap H^2(\Omega),\qquad \Big|\int_\Omega\mathcal{A}\nabla(\phi^\varepsilon-\phi_{g_\varepsilon})\nabla U\Big|\le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}||U||_{H^2(\Omega)}.$$

Now, let $U_\varepsilon\in H^1_0(\Omega)$ be the solution of the following variational problem:

$$\int_\Omega\mathcal{A}\nabla v\nabla U_\varepsilon=\int_\Omega v(\phi^\varepsilon-\phi_{g_\varepsilon}),\qquad\forall v\in H^1_0(\Omega).$$

Under the assumption on the boundary of Ω, we know that U_ε belongs to $H^1_0(\Omega)\cap H^2(\Omega)$ and satisfies $||U_\varepsilon||_{H^2(\Omega)}\le C||\phi^\varepsilon-\phi_{g_\varepsilon}||_{L^2(\Omega)}$ (the constant do not depend on ε). Therefore, the second estimate in (6.2) is proved.

Step 3. We prove the third estimate in (6.2) and (6.3). The partial derivative $\dfrac{\partial\phi_{g_\varepsilon}}{\partial x_i}$ satisfies

$$\mathrm{div}\Big(\mathcal{A}\nabla\big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\big)\Big)=0\qquad\text{in}\quad\Omega,\qquad\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\in L^2(\Omega).$$

Thus, from Remark 5.8 and estimate $(6.4)_2$ we get

$$\Big\|\rho\nabla\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)}\le C\Big\|\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big\|_{L^2(\Omega)}\le C||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.\tag{6.15}$$

Now, let U be in $H^1_0(\Omega)$, the function ρU belongs to $H^1_{1/\rho}(\Omega)$. Applying the Theorem 3.2 with the function ρU, there exists $\widehat{u}_\varepsilon\in L^2(\Omega;H^1_{per}(Y))$ such that

$$||\mathcal{T}_\varepsilon(\nabla(\rho U))-\nabla(\rho U)-\nabla_y\widehat{u}_\varepsilon||_{L^2(Y;(H^1_\rho(\Omega;\mathbb{R}^n))')}\le C\varepsilon||\rho U||_{H^1_{1/\rho}(\Omega)}\le C\varepsilon||U||_{H^1(\Omega)}.\tag{6.16}$$

The above estimates (6.15) and (6.16) lead to

$$\Big|\int_{\Omega\times Y}A\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\Big(\mathcal{T}_\varepsilon\big(\nabla(\rho U)\big)-\nabla(\rho U)-\nabla_y\widehat{u}_\varepsilon\Big)\Big|\le C\varepsilon||U||_{H^1(\Omega)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}$$

By definition of the correctors χ_i we have

$$\int_{\Omega\times Y}A\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\nabla_y\widehat{u}_\varepsilon=0.$$

Besides, from the definitions of the function ϕ_{g_ε} and the homogenized matrix $\mathcal{A}$ we have

$$0=\int_\Omega\mathcal{A}\nabla\phi_{g_\varepsilon}\nabla(\rho U)=\int_{\Omega\times Y}A\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\nabla(\rho U).$$

The above inequality and equalities yield

$$\Big|\int_{\Omega\times Y}A\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\mathcal{T}_\varepsilon\big(\nabla(\rho U)\big)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.\qquad(6.17)$$

We have

$$\nabla(\rho U)=\rho\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big).$$

Then since $U/\rho\in L^2(\Omega)$ and $||U/\rho||_{L^2(\Omega)}\le C||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}$ and due to (3.2) we get

$$\Big\|\mathcal{T}_\varepsilon\big(\nabla(\rho U)\big)-\rho\mathcal{T}_\varepsilon\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)}\le C\varepsilon\Big\|\nabla U+\nabla\rho\frac{U}{\rho}\Big\|_{L^2(\Omega;\mathbb{R}^n)}\le C\varepsilon||U||_{H^1(\Omega)}.$$

From (6.17) and the above inequalities we deduce that

$$\Big|\int_{\Omega\times Y}A\Big(\rho\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\rho\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\mathcal{T}_\varepsilon\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.$$

We recall that $\rho\nabla\phi_{g_\varepsilon}\in H^1_0(\Omega;\mathbb{R}^n)$, hence from $(2.14)_2$, $(2.17)_1$ and (6.15) we get

$$\begin{aligned}&\Big|\int_{\Omega\times Y}A\Big(\rho\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\rho\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\nabla_y\chi_i\Big)\mathcal{T}_\varepsilon\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\\&-\int_{\Omega\times Y}A\Big(\mathcal{T}_\varepsilon(\rho\nabla\phi_{g_\varepsilon})+\sum_{i=1}^n\mathcal{M}_\varepsilon\Big(\rho\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\nabla_y\chi_i\Big)\mathcal{T}_\varepsilon\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

Then transforming by inverse unfolding we obtain

$$\Big|\int_{\widehat{\Omega}_\varepsilon}A_\varepsilon\Big(\rho\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\mathcal{M}_\varepsilon\Big(\rho\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.$$

Now, thanks to (2.28) and (6.15) we get

$$\Big|\int_\Omega A_\varepsilon\rho\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\mathcal{M}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)\Big(\nabla U+\nabla\rho\frac{U}{\rho}\Big)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.$$

Then using $(2.29)_1$ it leads to

$$\Big|\int_\Omega A_\varepsilon\Big(\nabla\phi_{g_\varepsilon}+\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\nabla_y\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)\nabla(\rho U)\Big|\le C\varepsilon||\nabla U||_{L^2(\Omega;\mathbb{R}^n)}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.$$

We recall that $\displaystyle\int_\Omega A_\varepsilon\nabla\phi^\varepsilon\nabla(\rho U)=0$. We choose $U=\rho\Big(\phi^\varepsilon-\phi_{g_\varepsilon}-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\chi_i\big(\frac{\cdot}{\varepsilon}\big)\Big)$ which belongs to $H^1_0(\Omega)$. Due to the second estimate in (6.2), the third one in (6.2) follows immediately.
The estimate (6.3) is the consequence of $(2.29)_2$, $(6.2)_2$, $(6.2)_3$, $(6.4)_2$ and (6.15). $\square$

Corollary 6.2. *Let $\big(\phi^\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions belonging to $H^1(\Omega)$ and satisfying (6.1). We set $g_\varepsilon=\phi^\varepsilon_{|\partial\Omega}$, if we have*

$$g_\varepsilon\rightharpoonup g\quad\textit{weakly in}\quad H^{1/2}(\partial\Omega)$$

then we obtain

$$\begin{aligned}&\phi^\varepsilon\rightharpoonup\phi_g\quad\textit{weakly in}\quad H^1(\Omega),\\&\phi^\varepsilon-\phi_g-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_g}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\longrightarrow 0\quad\textit{strongly in}\quad H^1_\rho(\Omega).\end{aligned}\tag{6.18}$$

Moreover, if

$$g_\varepsilon\longrightarrow g\quad\textit{strongly in}\quad H^{1/2}(\partial\Omega)\tag{6.19}$$

then we have

$$\phi^\varepsilon-\phi_g-\varepsilon\sum_{i=1}^n\mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_g}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\longrightarrow 0\quad\textit{strongly in}\quad H^1(\Omega).\tag{6.20}$$

Proof. Thanks to $(6.2)_1$ the sequence $\big(\phi^\varepsilon\big)_{\varepsilon>0}$ is uniformly bounded in $H^1(\Omega)$. Then due to Lemma 5.1 and Remark 5.8 we get

$$||\phi_g-\phi_{g_\varepsilon}||_\rho\le C||g-g_\varepsilon||_{H^{-1/2}(\partial\Omega)}$$

which with $(6.2)_2$ (resp. $(6.2)_3$) give the convergence $(6.18)_1$ (resp. $(6.18)_2$).
Under the assumption (6.19), we use (5.2) and we proceed as in the proof of the Theorem 6.1 of [10] in order to obtain the strong convergence (6.20). $\square$

Theorem 6.3. *Let ϕ^ε be the solution of the following homogenization problem:*

$$-div\big(A_\varepsilon \nabla \phi^\varepsilon\big) = f \quad in \quad \Omega, \qquad \phi^\varepsilon = g \quad on \quad \partial\Omega$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. We have

$$||\phi^\varepsilon - \Phi||_{L^2(\Omega)} \le C\big\{\varepsilon||f||_{L^2(\Omega)} + \varepsilon^{1/2}||g||_{H^{1/2}(\partial\Omega)}\big\},$$

$$\Big\|\rho\Big(\nabla\phi^\varepsilon - \nabla\Phi - \sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\nabla_y\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)} \le C\big\{\varepsilon||f||_{L^2(\Omega)} + \varepsilon^{1/2}||g||_{H^{1/2}(\partial\Omega)}\big\}$$

where Φ is the solution of the homogenized problem

$$-div\big(\mathcal{A}\nabla\Phi\big) = f \quad in \quad \Omega, \qquad \Phi = g \quad on \quad \partial\Omega.$$

Moreover we have

$$\phi^\varepsilon - \Phi - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big) \longrightarrow 0 \quad strongly\ in \quad H^1(\Omega). \tag{6.21}$$

Proof. Let $\widetilde{\phi}^\varepsilon$ be the solution of the homogenization problem

$$\widetilde{\phi}^\varepsilon \in H^1_0(\Omega), \qquad -\mathrm{div}\big(A_\varepsilon\nabla\widetilde{\phi}^\varepsilon\big) = f \qquad \text{in} \quad \Omega$$

and $\widetilde{\Phi}$ the solution of the homogenized problem

$$\widetilde{\Phi} \in H^1_0(\Omega), \qquad -\mathrm{div}\big(\mathcal{A}\nabla\widetilde{\Phi}\big) = f \qquad \text{in} \quad \Omega.$$

The Theorem 3.2 in [16] gives the following estimate:

$$||\widetilde{\phi}^\varepsilon - \widetilde{\Phi}||_{L^2(\Omega)} + \Big\|\rho\nabla\Big(\widetilde{\phi}^\varepsilon - \widetilde{\Phi} - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\widetilde{\Phi}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big)\Big\|_{L^2(\Omega;\mathbb{R}^n)} \le C\varepsilon||f||_{L^2(\Omega)} \tag{6.22}$$

while the Theorem 4.1 in [15] gives

$$\Big\|\widetilde{\phi}^\varepsilon - \widetilde{\Phi} - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\widetilde{\Phi}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{H^1(\Omega)} \le C\varepsilon^{1/2}||f||_{L^2(\Omega)}. \tag{6.23}$$

The function $\phi^\varepsilon - \widetilde{\phi}^\varepsilon$ satisfies

$$\mathrm{div}\big(A_\varepsilon\nabla(\phi^\varepsilon - \widetilde{\phi}^\varepsilon)\big) = 0 \qquad \text{in} \quad \Omega, \qquad \phi^\varepsilon - \widetilde{\phi}^\varepsilon = g \qquad \text{on} \quad \partial\Omega.$$

Thanks to the inequalities (6.2) and (6.22) we deduce the estimates of the theorem. The strong convergence (6.21) is a consequence of (6.23) and the strong convergence (6.20) after having observed that $\Phi - \widetilde{\Phi} = \phi_g$. □

Remark 6.4. *In Theorem 6.3, if $g \in H^{3/2}(\partial\Omega)$ then in the estimates therein, we can replace $\varepsilon^{1/2}||g||_{H^{1/2}(\partial\Omega)}$ with $\varepsilon||g||_{H^{3/2}(\partial\Omega)}$. Moreover we have the following H^1-global error estimate:*

$$\Big\|\phi^\varepsilon - \Phi - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\Phi}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\big\{||f||_{L^2(\Omega)} + ||g||_{H^{3/2}(\partial\Omega)}\big\}.$$

7 A first result with strongly oscillating boundary data

In this section we consider the solution ϕ^ε of the homogenization problem

$$\begin{aligned} &\operatorname{div}\big(A_\varepsilon \nabla\phi^\varepsilon\big) = 0 \qquad \text{in} \quad \Omega \\ &\phi^\varepsilon = g_\varepsilon \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{7.1}$$

where $g_\varepsilon \in H^{1/2}(\partial\Omega)$. As a consequence of the Theorem 6.1 we obtain the following result:

Theorem 7.1. *Let ϕ^ε be the solution of the problem* (7.1)*. If we have*

$$g_\varepsilon \rightharpoonup g \quad \textit{weakly in} \quad H^{-1/2}(\partial\Omega)$$

and

$$\varepsilon^{1/2} g_\varepsilon \longrightarrow 0 \quad \textit{strongly in} \quad H^{1/2}(\partial\Omega) \tag{7.2}$$

then

$$\phi^\varepsilon \rightharpoonup \phi_g \quad \textit{weakly in} \quad H^1_\rho(\Omega). \tag{7.3}$$

Furthermore, if we have

$$g_\varepsilon \longrightarrow g \quad \textit{strongly in} \quad H^{-1/2}(\partial\Omega)$$

then

$$\phi^\varepsilon - \phi_g - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big) \longrightarrow 0 \quad \textit{strongly in} \quad H^1_\rho(\Omega). \tag{7.4}$$

Proof. Due to (6.3) the sequence $\big(\phi^\varepsilon\big)_{\varepsilon>0}$ is uniformly bounded in $H^1_\rho(\Omega)$. From the estimates $(6.2)_3$ and $(6.4)_2$ we get

$$\Big\|\phi^\varepsilon - \phi_{g_\varepsilon} - \varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{H^1_\rho(\Omega)} \le C\varepsilon^{1/2}||g_\varepsilon||_{H^{1/2}(\partial\Omega)}.$$

Then using the variational problem (5.5) and estimate $(6.4)_2$ we obtain

$$\phi_{g_\varepsilon} \rightharpoonup \phi_g \quad \text{weakly in} \quad H^1_\rho(\Omega).$$

Since the sequence $\varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big)$ is uniformly bounded in $H^1_\rho(\Omega)$ and strongly converges to 0 in $L^2(\Omega)$, we have $\varepsilon\sum_{i=1}^n \mathcal{Q}_\varepsilon\Big(\frac{\partial\phi_{g_\varepsilon}}{\partial x_i}\Big)\chi_i\Big(\frac{\cdot}{\varepsilon}\Big) \rightharpoonup 0$ weakly in $H^1_\rho(\Omega)$. Therefore the weak convergence (7.3) is proved.

In the case $g_\varepsilon \longrightarrow g$ strongly in $H^{-1/2}(\partial\Omega)$, the estimates (5.4) and (5.8) lead to

$$||\phi_{g_\varepsilon} - \phi_g||_{H^1_\rho(\Omega)} \le C||g_\varepsilon - g||_{H^{-1/2}(\partial\Omega)}.$$

Hence with $(2.29)_2$ they yield (7.4). ∎

In a forthcoming paper we will show that in both cases (weak or strong convergence of the sequence $(g_\varepsilon)_{\varepsilon>0}$ towards g in $H^{-1/2}(\partial\Omega)$) the assumption (7.2) is essential in order to obtain at least (7.3).

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